Math 548 | **Geometric Structures** | **Winter 1995**

ASSIGNMENT #1

DUE 1/27/95

Reading: do Carmo, Chapter 9.

I. Practice problems.

1. Let (M, g) be a Riemannian manifold, $p, q \in M$, $c \in \Omega_{pq}$, and $\{c_s\}$ a proper variation of c. Assuming that c has nowhere-vanishing tangent vector, compute the first and second variations of the length functional $L(c_s)$.
2. If you haven't already done so, prove that a connected non-orientable manifold has an orientable double cover (do Carmo, p. 34 #12).

II. Required problems.

3. do Carmo, p. 207 #1.
4. do Carmo, p. 208 #4.
5. do Carmo, p. 208 #5.
6. do Carmo, p. 208 #6.

III. Optional problems.

7. do Carmo, p. 207 #2.
8. do Carmo, p. 207 #3.
9. Suppose (M, g) is a Riemannian manifold, and $S \subset M$ is a compact, embedded submanifold with or without boundary. Let $NS \to S$ be the 'normal bundle' of S, i.e. the subset $\bigcup_{p\in S}(T_pS)^\perp$ of TM.
   (a) Show that NS is a smooth submanifold of TM (possibly with boundary) whose dimension is the same as that of M.
   (b) For $\varepsilon > 0$, let $N_\varepsilon S$ denote the set of vectors in NS of length less than ε. Prove the following 'tubular neighborhood theorem': *There exists $\varepsilon > 0$ such that the restriction of the exponential map is an embedding of $N_\varepsilon S$ into M, whose image is a neighborhood of S in M.*
10. Let (M, g) be a Riemannian manifold and $p, q \in M$. For any curve $\gamma \in \Omega_{pq}$, on each interval $[t_i, t_{i+1}]$ where γ is smooth, let $\tilde\gamma^{(i)} \colon [t_i, t_{i+1}] \to TM$ be the lift to TM, defined by $\tilde\gamma^{(i)}(t) = (\gamma(t), \dot\gamma(t))$, where $\dot\gamma$ is interpreted as a one-sided derivative at the endpoints. Let $\tilde g$ be any Riemannian metric on TM, and define a distance function on Ω_{pq} by
   $$\tilde d(\gamma_1, \gamma_2) = \sup\{d_{\tilde g}(\tilde\gamma_1^{(i)}(t), \tilde\gamma_2^{(i)}(t)) : t \in [t_i, t_{i+1}], 0 \le i \le k\}.$$
   Suppose $\gamma \in \Omega_{pq}$ is a smooth, embedded curve. Using the result of Problem 9b, show that there exists a $\tilde d$-neighborhood $\mathfrak{U}$ of γ in Ω_{pq} such that any curve $c \in \mathfrak{U}$ can be written $c(t) = \gamma_{\varepsilon/2}(t)$ for some proper variation $\{\gamma_s\}_{|s|<\varepsilon}$ of γ.