Math 547 **Geometric Structures** **Autumn 1994**

ASSIGNMENT #5
DUE 12/9/94

Reading: do Carmo, Section 5.3, Chapter 7, Sections 8.1–8.4.

I. Practice problems.

1. do Carmo, p. 123 #8.
2. do Carmo, p. 152 #1, 2, 3.
3. do Carmo, p. 154 #12.
4. We now have two kinds of "metrics" on a Riemannian manifold—the Riemannian metric and the distance function. Correspondingly, there are two definitions of "isometry" between Riemannian manifolds—namely, a map that preserves the Riemannian metric or a map that preserves the distance function. Prove that these two definitions are equivalent.

II. Required problems.

5. Let (M, g) be a Riemannian manifold and $p \in M$. Let $(x^1, \ldots, x^n)$ be any Riemannian normal coordinates centered at p. Writing the second-order Taylor series of g_{ij} in these coordinates as

$$g_{ij}(x) = a_{ij} + \sum_k b_{ijk} x^k + \sum_{kl} c_{ijkl} x^k x^l + O(|x|^3),$$

compute the coefficients a_{ij}, b_{ijk}, and c_{ijkl}. [Hint: use the Taylor series for lengths of Jacobi fields derived in class.

6. do Carmo, p. 153 #5. [The last word should be *infinite*, not *unbounded*.]
7. If M is a simply-connected Riemannian manifold with non-positive sectional curvature, show that "two points determine a line," i.e. if p and q are distinct points in M there exists an embedded geodesic $\gamma\colon \mathbf{R} \to M$ whose image contains p and q, and which is unique up to reparametrization.

III. Optional problems.

8. do Carmo, p. 152 #4.
9. Suppose M is an n-dimensional Riemannian manifold, and a Lie group G acts isometrically on M. Show that $\dim G \leq n(n+1)/2$, with equality only if M has constant sectional curvature.