Math 547 **Geometric Structures** **Autumn 1994**

ASSIGNMENT #4

DUE 12/2/94

Reading: do Carmo, Sections 4.4, 5.1, 5.2; for the Gauss-Bonnet Theorem, look at Spivak Volume 2, Chapter 6 or any text on differential geometry of curves and surfaces.

I. Practice problems.

1. On a manifold M, let $\mathcal{T}^r_s(M)$ denote the bundle of tensors of rank $\binom{r}{s}$ on M, and $\mathcal{X}^r_s(M)$ the space of sections of $\mathcal{T}^r_s(M)$, i.e. the space of $\binom{r}{s}$ tensor fields. By definition, a *connection* on $\mathcal{T}^r_s$ is a map $\nabla \colon \mathcal{X}(M) \times \mathcal{X}^r_s(M) \to \mathcal{X}^r_s(M)$ which satisfies
   - $\nabla_X T$ is linear over $C^\infty(M)$ in X;
   - $\nabla_X T$ is linear over $\mathbf{R}$ in T;
   - $\nabla_X(fT) = (Xf)T + f\nabla_X T$ for $f \in C^\infty(M)$.

   (a) If ∇ is any affine connection on M, show that there exists a unique connection (also denoted ∇) on $T^*M = \mathcal{T}^1_0(M)$ which satisfies the following "product rule" for any 1-form ω and vector field Y:
   $$X(\omega(Y)) = (\nabla_X\omega)(Y) + \omega(\nabla_X Y).$$

   (b) Show that ∇ extends uniquely to a connection on each tensor bundle $\mathcal{T}^r_s(M)$ which satisfies the product rule:
   $$\begin{aligned} X(T(\omega^1,\dots,\omega^s,Y_1,\dots,Y_r)) = {} & (\nabla_X T)(\omega^1,\dots,\omega^s,Y_1,\dots,Y_r) \\ & + \sum_{i=1}^{s} T(\omega^1,\dots,\nabla_X\omega^i,\dots,\omega^s,Y_1,\dots,Y_r) \\ & + \sum_{i=1}^{r} T(\omega^1,\dots,\omega^s,Y_1,\dots,\nabla_X Y_i,\dots,Y_r), \end{aligned}$$
   where ω^i are 1-forms and Y_i are vector fields.

   (c) Show that ∇ satisfies the product rule with respect to tensor products:
   $$\nabla_X(T\otimes S) = \nabla_X T\otimes S + T\otimes\nabla_X S.$$

   (d) For any $\binom{r}{s}$ tensor field T, show that the formula
   $$\nabla T(\omega^1,\dots,\omega^s,Y_1,\dots,Y_r,X) = (\nabla_X T)(\omega^1,\dots,\omega^s,Y_1,\dots,Y_r)$$
   defines an $\binom{r+1}{s}$ tensor field ∇T, called the *total covariant derivative* or *covariant differential* of T. [Cf. the discussion in do Carmo, Chapter 4, §5.]

   (e) If g is a Riemannian metric on M and ∇ is any connection, show that ∇ is compatible with g if and only if $\nabla g = 0$.

2. Let (M, g) be a Riemannian manifold, $\{E_1, \ldots, E_n\}$ any local frame for M, and $\{\varphi^1, \ldots, \varphi^n\}$ its dual coframe.

   (a) If $\omega = \sum \omega_i \varphi^i$ is a 1-form, and $\sum \omega_{i;j} \varphi^i \otimes \varphi^j$ is the local expression for $\nabla\omega$ (see Problem 1), show that

   $$\omega_{i;j} = E_j(\omega_i) - \sum_k \Gamma^k_{ji} \omega_k.$$

   (b) Derive a similar formula for ∇T, where T is any tensor field of rank $\binom{r}{s}$.

3. do Carmo, p. 106 #7 (if you didn't do it already in the course of doing #8).

4. Two metrics g_1 and g_2 are said to be *conformal* if $g_2 = f g_1$ for some positive function f. A diffeomorphism $F\colon (M_1, g_1) \to (M_2, g_2)$ is said to be conformal if $f^* g_2$ is conformal to g_1. [This differs slightly from the usual definition used in complex analysis—a diffeomorphism between regions in the complex plane is conformal in the sense of complex analysis iff it is conformal in our sense and orientation-preserving.] Let $\sigma\colon S^2 \setminus \{(0,0,1)\} \to \mathbf{R}^2$ denote the stereographic projection from the North Pole (see do Carmo, p. 19). Show that σ is conformal.

5. do Carmo, p. 119 #1.

II. Required problems.

6. do Carmo, p. 108 #10.

7. (a) If u is a function, define $\nabla u = du$, and $\nabla^2 u = \nabla(\nabla u)$. Show that $\nabla^2 u$ is a symmetric 2-tensor field, called the *covariant Hessian* of u. [See do Carmo, p. 141, Problem 11 for a different definition.]

   (b) If ω is a 1-form, let

   $$\sum_{ijk} \omega_{i;jk} \varphi^i \otimes \varphi^j \otimes \varphi^k$$

   be the local expression for $\nabla^2\omega$. Prove the *Ricci identity*

   $$\omega_{i;jk} - \omega_{i;kj} = \sum_l R^l_{kji} \omega_l.$$

   [Hint: Instead of expanding out the components of $\omega_{i;jk}$ using the formulas of Problem 2, try using the definitions of the curvature operator and the total covariant derivative, and the product rules of Problem 1.]

8. Let $M \subset \mathbf{R}^3$ be a compact, oriented, embedded surface.

   (a) Show that there is a point on M where $K > 0$. [Hint: consider a point $p \in M$ at a maximum distance from the origin, and use a refinement of the proof of Problem 8, Assignment 3.]

   (b) If M is not diffeomorphic to S^2, show that there is also a point where $K < 0$.

9. Compute the Gaussian curvature of the hyperbolic plane $\mathbf{H}^2$.

10. do Carmo, p. 119 #2.

III. Optional problems.

11. do Carmo, p. 107 #9.
12. Let M be a compact 2-manifold. Show that M has a smooth triangulation.
13. Let $\gamma(v) = (f(v), g(v))$ be a smooth curve in the (x, z) plane, and let M be the surface obtained by revolving γ around the z-axis, parametrized as in do Carmo, p. 77 #1.
    (a) Assuming that γ is unit-speed, compute the Gaussian curvature of M as a function of u and v.
    (b) Show that there exists a 1-parameter family of surfaces of revolution $\{M_t, t \in (-\varepsilon, \varepsilon)\}$ with constant Gaussian curvature $+1$, with M_0 equal to the standard unit sphere. Describe them geometrically. Can you produce them with a hollow rubber ball?