Math 547 **Geometric Structures** **Autumn 1994**

Assignment #3

Due 11/18/94

Reading: do Carmo, Chapter 4 (skip 4.4) and Chapter 6.

I. Practice problems.

1. If M and N are manifolds with M compact and connected, any local diffeomorphism $M \to N$ is a covering map.
2. do Carmo, p. 139 #1.
3. do Carmo, p. 139 #2, 5.
4. Let ∇ be an affine connection on a manifold M. Define a map $\tau\colon \mathfrak{X}(M)\times\mathfrak{X}(M) \to \mathfrak{X}(M)$ by
$$\tau(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y].$$
Show that τ defines a $\binom{2}{1}$-tensor field on M, called the *torsion tensor field*, which vanishes iff ∇ is symmetric.
5. Let
$$X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$
Show that X, Y, Z form a basis for the Lie algebra $\mathfrak{so}(3)$ of $SO(3)$, and compute their Lie brackets.

II. Required problems.

6. do Carmo, p. 106 #8.
7. Suppose Ω is an open set in $\mathbf{R}^n$ and $f\colon \Omega \to \mathbf{R}$ is a smooth map. Let $M = \{(x, f(x)) : x \in \Omega\} \subset \mathbf{R}^{n+1}$ be the graph of f. Observe that the map $\varphi\colon \Omega \to M$ given by $\varphi(u) = (u, f(u))$ gives a global parametrization of M; the corresponding coordinates $(u^1, \dots, u^n)$ are called *graph coordinates*.
   (a) Let N be the upward-pointing unit normal vector field along M. Compute the components of the second fundamental form B_N in graph coordinates, in terms of f and its partial derivatives.
   (b) Compute the Gaussian and mean curvatures of M in terms of f.
   (c) If $M \subset \mathbf{R}^{n+1}$ is any isometrically immersed hypersurface, and p any point in M, show that it is possible to choose Euclidean coordinates such that M is locally given by a graph of the form $x^{n+1} = f(x^1, \dots, x^n)$, where the second-order Taylor series of f is the quadratic form
$$Q(x,y) = \sum_i \lambda_i (x^i)^2$$
where $\{\lambda_i\}$ are the principal curvatures at p.

8. Suppose $M \subset \mathbf{R}^3$ is a compact, embedded surface. Show that M cannot have strictly negative Gaussian or mean curvature. [Hint: consider a point where the z-coordinate takes on its minimum value, and use problem 7c.]

9. Let $M \subset \mathbf{R}^3$ be an isometrically immersed surface, $p \in M$, and $\lambda_1 \leq \lambda_2$ the principal curvatures of M at p with respect to some unit normal N.

   (a) Show that λ_1 and λ_2 are the minimum and maximum values of $II_N(X)$ over unit vectors X tangent to M.

   (b) Show that, if X is a unit vector tangent to M, $II_N(X)$ is the signed curvature at p of the plane curve obtained by intersecting M with the plane spanned by X and N, with a positive sign meaning curving toward N, and a negative sign curving away.

10. Suppose G is a Lie group with Lie algebra $\mathfrak{g}$. For any left-invariant vector fields $(X_1, \ldots, X_n)$ that form a basis for $\mathfrak{g}$, define the *structure constants* c_{ij}^k by

$$[X_i, X_j] = \sum_k c_{ij}^k X_k.$$

   (a) Show that a metric g on G is left-invariant if and only if $g_{ij} = g(X_i, X_j)$ are constants.

   (b) For an arbitrary left-invariant metric g, compute the Christoffel symbols of the Levi-Civita connection (with respect to the basis $\{X_i\}$) in terms of c_{ij}^k and g_{ij}.

   (c) For any positive constant a, define a left-invariant metric on $SO(3)$ by letting X, Y, aZ be an orthonormal frame, where X, Y, Z are as in Problem 5. Compute the sectional curvatures $K(X,Y)$, $K(Y,Z)$, and $K(Z,X)$ of this metric. [The case $a = 1$ is a familiar Riemannian manifold—do you recognize it?]

III. Optional problems.

11. Let ∇ be an affine connection on a manifold M.

   (a) Show that the Christoffel symbols are not the components of a tensor field on M, that is, there is no $\binom{2}{1}$-tensor field Γ such that, if $(X_1, \ldots, X_n)$ is any local frame for M and Γ_{ij}^k are its Christoffel symbols, then

$$\Gamma(X_i, X_j) = \sum_k \Gamma_{ij}^k X_k.$$

   (b) If ∇ and $\widetilde{\nabla}$ are any two connections on M, show that the difference between them defines a $\binom{2}{1}$-tensor field by

$$D(X,Y) = \nabla_X Y - \widetilde{\nabla}_X Y,$$

   called the *difference tensor*.

   (c) Show that ∇ and $\widetilde{\nabla}$ have the same geodesics if and only if the difference tensor is antisymmetric, i.e. $D(X,Y) = -D(Y,X)$. [Hint: the geodesic equation involves only expressions of the form $\nabla_X X$.]

12. Let ∇ be an affine connection on a manifold M. We can define the *curvature operator* of ∇ by the same formula as in the Riemannian case; ∇ is said to be *flat* if $R(X,Y)Z \equiv 0$. Prove that the following are equivalent.

   (a) ∇ is flat.

   (b) Near every point $p \in M$, there exists a local frame which is parallel along every curve.

   (c) For all $p, q \in M$, parallel translation along a curve segment γ from p to q depends only on the homotopy class of γ.

   (d) For all sufficiently nearby points $p, q \in M$, parallel translation from p to q depends only on p and q, not on the curve segment joining them.

   (e) Parallel translation around any sufficiently small closed curve is the identity.

   [See do Carmo, p. 104, #4 for one idea of how to approach this; but an easier proof is available using ideas developed in lecture.]