Math 547 **Geometric Structures** **Autumn 1994**

ASSIGNMENT #2

DUE 11/4/94

Reading: do Carmo, Chapter 3.

In all of the following problems, unless specified otherwise, M is a smooth Riemannian n-manifold with metric g.

I. Practice problems.

1. Suppose (M,g) and $(\widetilde{M},\tilde{g})$ are Riemannian manifolds, and $\varphi\colon M\to\widetilde{M}$ is an isometry, i.e. a diffeomorphism such that $\varphi^*\tilde{g}=g$.
   (a) Show that φ takes the Riemannian connection ∇ of M to the Riemannian connection $\widetilde{\nabla}$ of $\widetilde{M}$, that is if $X,Y\in\mathfrak{X}(M)$, then
   $$\varphi_*(\nabla_X Y)=\widetilde{\nabla}_{\varphi_* X}(\varphi_* Y).$$
   (b) Show that φ takes geodesics of M to geodesics of N.
2. If you didn't already prove it in the course of doing the last homework assignment, prove that if M is a connected Riemannian manifold, $\varphi\colon M\to M$ is an isometry that fixes a point $p\in M$, and $\varphi_*=id\colon T_pM\to T_pM$, then φ is the identity map.
3. Show that the *unit tangent bundle* $SM=\{(x,v)\in TM,\ |v|_g=1\}$ is a regular submanifold of TM of codimension 1.
4. Let $c\colon[a,b]\to M$ be a smooth curve segment. Show that the length
   $$\ell(c)=\int_a^b|\dot{c}(t)|\,dt$$
   is independent of parametrization, i.e. if $u\colon[\tilde{a},\tilde{b}]\to[a,b]$ is a diffeomorphism and $\tilde{c}=c\circ u$, then $\ell(\tilde{c})=\ell(c)$. Extend this to piecewise smooth curve segments and piecewise smooth reparametrizations.

II. Required problems.

5. do Carmo, p. 77 #1.
6. do Carmo, p. 81 #4.
7. do Carmo, p. 81 #5.
8. do Carmo, p. 83 #7.
9. Consider $\mathbf{R}^n$ as a Riemannian manifold with the Euclidean metric.
   (a) Let G be the set of $(n+1)\times(n+1)$ real matrices of the form
   $$\begin{pmatrix}A & b\\ 0 & 1\end{pmatrix},$$
   where $A\in O(n)$ and $b\in\mathbf{R}^n$. Show that G is a closed Lie subgroup of $GL(n+1,\mathbf{R})$.

(b) Define a map $G \times \mathbf{R}^n \to \mathbf{R}^n$ by identifying $\mathbf{R}^n$ with the subset

$$S = \{(x,1) \in \mathbf{R}^{n+1} : x \in \mathbf{R}^n\}$$

of $\mathbf{R}^{n+1}$ and restricting the linear action of G on $\mathbf{R}^{n+1}$ to S. Show that this is a smooth action of G on $\mathbf{R}^n$ by isometries of the Euclidean metric.

(c) Using Problem (2), show that all isometries of $\mathbf{R}^n$ are induced in this way by elements of G.

(d) Strengthen the result above by showing that if U, V are connected open subsets of $\mathbf{R}^n$ and $\varphi\colon U \to V$ is an isometry, then φ is the restriction to U of an element of G.

10. Let $\mathbf{H}^2$ denote the *hyperbolic plane*, i.e. the upper half-plane $\{(x,y) : y > 0\}$ with the metric g of Assignment 1, Problem 11. Let $SL(2,\mathbf{R})$ denote the group of 2×2 matrices with determinant 1.

    (a) Considering $\mathbf{H}^2$ as a subset of the complex plane with coordinate $z = x + iy$, let

    $$A \cdot z = \frac{az+b}{cz+d}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbf{R}).$$

    Show that this defines a smooth action of $SL(2,\mathbf{R})$ on $\mathbf{H}^2$ by isometries of the hyperbolic metric, and that $\mathbf{H}^2$ is a Riemannian homogeneous space.

    (b) Determine all isometries of $\mathbf{H}^2$.

    (c) Show that the geodesics of $\mathbf{H}^2$ are exactly the vertical half-lines and the semicircles intersecting the x-axis orthogonally (suitably parametrized).

    (d) Determine the unit-speed parametrization of the geodesic whose image is the y-axis.

III. Optional problems.

11. do Carmo, p. 78 #2.