Math 547 | **Geometric Structures** | **Autumn 1994**

ASSIGNMENT #1

DUE 10/21/94

Reading: do Carmo, Chapters 0, 1, and 2.

I. Practice problems.

1. do Carmo, p. 32 #3, 4, 5, 6.
2. do Carmo, p. 45 #1, 2, 3.
3. Prove the following *Escape Lemma*: Let Y be a vector field on a manifold M, and $c\colon (\alpha,\beta)\to M$ an integral curve of Y. If $\beta<\infty$ and the image of c is contained in some compact subset $K\subset M$, then c extends to an integral curve on $(\alpha,\beta+\varepsilon)$ for some $\varepsilon>0$. [Cf. Boothby, Chapter 4, p. 139, Lemma 5.1.]
4. If $\{\Gamma^k_{ij},\ 1\le i,j,k\le n\}$ are n^3 arbitrary smooth functions on an open subset of $\mathbf{R^n}$, prove in detail that the rule

$$\nabla_X Y=\Bigl(XY^k+\sum_{i,j}\Gamma^k_{ij}X^iY^j\Bigr)\frac{\partial}{\partial x^k}$$

defines a connection.

5. do Carmo, p. 58 #9.

II. Required problems.

6. do Carmo, p. 46 #5.
7. do Carmo, p. 56 #1, 2.
8. do Carmo, p. 57 #3, 4. (This gives a very concrete geometric interpretation of the covariant derivative in the case of a surface in $\mathbf{R^3}$.)
9. do Carmo, p. 57 #5.
10. do Carmo, p. 58 #7.
11. (do Carmo, p. 58 #8, slightly paraphrased). [Note: do Carmo gives the final answers, and an additional hint for part (b). Avoid reading his answers and hints unless you are really stuck. Students in previous years have complained that do Carmo gave too many hints, thus stifling learning.] Let M be the upper half-plane $\{(x,y):y>0\}$ with the metric

$$g=\frac{dx^2+dy^2}{y^2},$$

i.e. the metric with $g_{11}=g_{22}=y^{-2}$, $g_{12}=g_{21}=0$ in Euclidean coordinates.

(a) Compute the Christoffel symbols of g in these coordinates.

(b) Parallel translate the vector $V_0 = \partial/\partial y$ at $(0,1)$ along the curve $(x,y) = (t,1)$ to obtain a vector field $V(t)$ along the curve. Find the angle $V(t)$ makes with the y-axis (with respect to either the Euclidean metric or g; why does it not matter which?).

III. Optional problems.

12. do Carmo, p. 34 #12.

13. Let $I \subset \mathbf{R}$ be an open interval, and for $1 \le j, k \le n$ let $A_j^k \colon I \to \mathbf{R}$ be arbitrary smooth functions. Show that the linear initial-value problem

$$\begin{aligned} \dot V^k(t) &= \sum_j A_j^k(t) V^j(t), \\ V^k(t_0) &= B^k \end{aligned} \tag{$*$}$$

has a unique smooth solution on all of I for any $t_0 \in I$ and any initial vector $(B^1, \ldots, B^n) \in \mathbf{R^n}$.

[Hint: Consider the vector field Y on the manifold $I \times \mathbf{R^n}$ given by

$$\begin{aligned} Y^0(x^0, \ldots, x^n) &= 1, \\ Y^k(x^0, \ldots, x^n) &= \sum_j A_j^k(x^0) x^j, \quad k = 1, \ldots, n. \end{aligned}$$

(a) Show that the solutions of $(*)$ are exactly the projections of the integral curves of Y to $\mathbf{R^n}$. (b) If $V(t) = (V^1(t), \ldots, V^n(t))$ is any solution to $(*)$, show that $d/dt(e^{-Ct}|V(t)|^2) \le 0$ for some positive constant C. (Here $|V(t)|$ is just the Euclidean norm.) (c) If an integral curve of Y is defined only on some proper subinterval of I, use Problem 3 above to derive a contradiction.]