Math 545 **Topology and Geometry of Manifolds** **Winter 2000**

Assignment #9

Due 3/10/2000

I. Required problems.

1. Let $V, W, X \in \mathcal{T}(M)$ and $f, g \in C^\infty(M)$. Show that
   (a) $[fV, gW] = fg[V, W] + f(Vg)W - g(Wf)V$.
   (b) $\mathcal{L}_V(fW) = (Vf)W + f\mathcal{L}_V W$.
   (c) $\mathcal{L}_{[V,W]}X = \mathcal{L}_V\mathcal{L}_W X - \mathcal{L}_W\mathcal{L}_V X$.

   (The second equality can be interpreted as a "product rule" for the Lie derivative.)
2. Let V and W be the vector fields of Exercise 6.2(b) in the notes. Compute the flows θ, ψ of V and W, and verify that they do not commute by finding explicit times s and t such that $\theta_s \circ \psi_t \neq \psi_t \circ \theta_s$.
3. Give an example of vector fields V, $\widetilde{V}$, and W on a manifold such that $V_p = \widetilde{V}_p$ but $(\mathcal{L}_V W)_p \neq (\mathcal{L}_{\widetilde{V}} W)_p$. (This shows that it is really necessary to know the vector field V to compute $(\mathcal{L}_V W)_p$, not just the vector V_p.)

II. Optional problems.

4. The following problem generalizes problem 2 from the last assignment. Suppose $F\colon M \longrightarrow N$ is a smooth map, $V \in \mathcal{T}(M)$, and $W \in \mathcal{T}(N)$, and let θ be the flow of V and ψ the flow of W. Show that V and W are F-related if and only if for each $t \in \mathbb{R}$, $\psi_t \circ F = F \circ \theta_t$ wherever either side is defined:

$$\begin{array}{ccc} M & \xrightarrow{\;F\;} & N \\ {\scriptstyle \theta_t}\big\downarrow & & \big\downarrow{\scriptstyle \psi_t} \\ M & \xrightarrow[\;F\;]{} & N \end{array}$$