Math 546 **Topology and Geometry of Manifolds** **Spring 2000**

Assignment #9

Due 5/26/2000

I. Required problems.

1. Let (M, g) be an oriented Riemannian n-manifold.
   (a) Define maps $\alpha\colon C^\infty(M) \longrightarrow \mathcal{A}^n(M)$ and $\beta\colon \mathfrak{T}(M) \longrightarrow \mathcal{A}^{n-1}(M)$ by
   $$\alpha(f) = f\,dV_g,$$
   $$\beta(X) = i_X dV_g.$$
   Show that α and β are isomorphisms (considering the domain and range as real vector spaces in each case).
   (b) Define the *divergence operator* $\operatorname{div}\colon \mathfrak{T}(M) \longrightarrow C^\infty(M)$ by
   $$\operatorname{div} = \alpha^{-1} \circ d \circ \beta,$$
   or equivalently,
   $$d(i_X dV_g) = (\operatorname{div} X) dV_g.$$
   Show that the divergence operator satisfies the following product rule for $f \in C^\infty(M)$, $X \in \mathfrak{T}(M)$:
   $$\operatorname{div}(fX) = f \operatorname{div} X + \langle \operatorname{grad} f, X\rangle.$$
   (c) On $\mathbb{R}^n$ with the Euclidean metric, show that
   $$\operatorname{div}\left(X^i \frac{\partial}{\partial x^i}\right) = \frac{\partial X^i}{\partial x^i}.$$
2. Let (M, g) be an oriented Riemannian 3-manifold. Define the *curl operator* $\operatorname{curl}\colon \mathfrak{T}(M) \longrightarrow \mathfrak{T}(M)$ by
   $$\operatorname{curl} X = \beta^{-1} d(X^\flat),$$
   or equivalently,
   $$i_{\operatorname{curl} X} dV_g = d(X^\flat).$$
   (a) Show that $\operatorname{curl} \circ \operatorname{grad} \equiv 0$ and $\operatorname{div} \circ \operatorname{curl} \equiv 0$.
   (b) On $\mathbb{R}^3$ with the Euclidean metric, show that
   $$\operatorname{curl}\left(P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + R\frac{\partial}{\partial z}\right) = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\frac{\partial}{\partial x} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\frac{\partial}{\partial y} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\frac{\partial}{\partial z}.$$

[The setup of Problems 1 and 2 can be summarized by the following commutative diagram:

$$\begin{array}{ccccccc}
C^\infty(M) & \xrightarrow{\text{grad}} & \mathfrak{T}(M) & \xrightarrow{\text{curl}} & \mathfrak{T}(M) & \xrightarrow{\text{div}} & C^\infty(M) \\
\downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle \flat} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \alpha} \\
C^\infty(M) & \xrightarrow{d} & \mathcal{A}^1(M) & \xrightarrow{d} & \mathcal{A}^2(M) & \xrightarrow{d} & \mathcal{A}^3(M),
\end{array}$$

in which the composition of any two consecutive horizontal maps is zero.]

3. Let (M, g) be an oriented Riemannian manifold, and let $\widetilde{M} \subset M$ be an immersed hypersurface. Suppose N is a smooth unit normal vector field along $\widetilde{M}$ (that is, a smooth unit-length vector field along $\widetilde{M}$ that is orthogonal to $T_p\widetilde{M}$ for each $p \in \widetilde{M}$). Let $\widetilde{g} = g|_{\widetilde{M}}$ denote the induced metric on $\widetilde{M}$, and let $dV_{\widetilde{g}}$ denote the Riemannian volume form on $\widetilde{M}$ with respect to the orientation induced by N. Show that $dV_{\widetilde{g}} = i_N dV_g$.

4. Let M be a smooth manifold and ω a 1-form on M. Show that for any smooth vector fields X, Y,

$$d\omega(X, Y) = \tfrac{1}{2}\big(X(\omega(Y)) - Y(\omega(X)) - \omega[X, Y]\big).$$

II. Optional problems.

5. Generalize the result of Problem 4 to a k-form ω by showing that

$$d\omega(X_1, \ldots, X_{k+1}) = \frac{1}{k+1}\Bigg(\sum_{1 \le i \le k+1} (-1)^{i-1} X_i(\omega(X_1, \ldots, \widehat{X}_i, \ldots, X_{k+1})) + \sum_{1 \le i < j \le k+1} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \widehat{X}_i, \ldots, \widehat{X}_j, \ldots, X_{k+1}) \Bigg),$$

where the hats indicate omitted arguments.

6. Let M be the open Möbius band, which is the quotient of $\mathbb{R} \times (-1, 1)$ by the action of the discrete group $\mathbb{Z}$ defined by $n \cdot (x, y) = (x + 1, -y)$. Show that M is not orientable.

7. Show that the projective plane $\mathbb{P}^2$ (considered as the quotient of $\mathbb{S}^2$ by the antipodal action of $\mathbb{Z}/\langle 2 \rangle$) is not orientable.