Math 545 **Topology and Geometry of Manifolds** **Winter 2000**

Assignment #8

Due 3/3/2000

I. Required problems.

1. If $N \subset M$ is an embedded submanifold and $\gamma\colon J \longrightarrow M$ is a smooth curve whose image happens to lie in N, show that $\gamma'(t)$ is in the subspace $T_{\gamma(t)}N$ of $T_{\gamma(t)}M$ for all $t \in J$. Give a counterexample if N is not embedded.
2. Let $F\colon M \longrightarrow M$ be a diffeomorphism, let V be a smooth vector field on M, and let θ be the flow generated by V. Show that V is invariant under F if and only if θ commutes with F, in the sense that $\theta_t \circ F = F \circ \theta_t$ whenever either side is defined. [Hint: use uniqueness of integral curves.]
3. Prove Lemma 5.12.
4. If V is a vector field on a smooth manifold M, the *support* of V is defined to be the closure of the set $\{p \in M : V_p \neq 0\}$. Show that every smooth vector field with compact support is complete.

II. Optional problems.

5. All the systems of differential equations that we have considered have been of the form

$$\gamma^{i\prime}(t) = V^i(\gamma(t)),$$

in which the functions V^i do not depend explicitly on the independent variable t. (Such a system is said to be *autonomous.*) If instead V is a function of (t, x) in some subset of $\mathbb{R} \times \mathbb{R}^n$, the resulting system is called *nonautonomous*; it can be thought of as a "time-dependent vector field" on a subset of $\mathbb{R}^n$. This problem shows that local existence, uniqueness, and smoothness for a nonautonomous system follow from the corresponding results for autonomous ones.

Suppose $U \subset \mathbb{R}^n$ is an open set, $J \subset \mathbb{R}$ is an open interval, and $V\colon J \times U \longrightarrow \mathbb{R}^n$ is a smooth map. For any $(t_0, x_0) \in J \times U$ and any sufficiently small $\varepsilon > 0$, show that there exists a neighborhood U_0 of x_0 in U and a smooth map $\theta\colon (t_0 - \varepsilon, t_0 + \varepsilon) \times U_0 \longrightarrow U$ such that for each $x \in U_0$, the curve $\gamma(t) = \theta(t, x)$ is the unique solution on $(t_0 - \varepsilon, t_0 + \varepsilon)$ to the nonautonomous initial-value problem

$$\begin{aligned} \gamma^{i\prime}(t) &= V^i(t, \gamma(t)), \\ \gamma^i(t_0) &= x^i. \end{aligned}$$

[Hint: replace this system of ODEs by an autonomous system in $\mathbb{R}^{n+1}$.]