## I. Required problems.

- 1. Let V be an n-dimensional vector space.
  - (a) Show that covectors  $\omega^1, \ldots, \omega^k$  on V are linearly dependent if and only if  $\omega^1 \wedge \cdots \wedge \omega^k = 0$ .
  - (b) Suppose  $\{\omega^1, \ldots, \omega^k\}$  and  $\{\eta^1, \ldots, \eta^k\}$  are two collections of independent covectors on V. Show that the collections have the same span if and only if for some nonzero constant c,

$$\omega^1 \wedge \dots \wedge \omega^k = c \, \eta^1 \wedge \dots \wedge \eta^k$$

- (c) If  $\omega \in \Lambda^k V$ ,  $\omega$  is said to be *decomposable* if  $\omega$  can be written  $\omega = \sigma^1 \wedge \cdots \wedge \sigma^k$ , where each  $\sigma^i$  is a covector. Is every 2-covector on V decomposable? Your answer will depend on the dimension of V. Give proof or counterexample.
- 2. Prove Lemma 11.10.
- 3. Define a 2-form  $\Omega$  on  $\mathbb{R}^3$  by

$$\Omega = x \, dy \wedge dz + y \, dz \wedge dx + z \, dx \wedge dy.$$

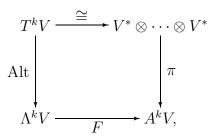
- (a) Compute  $\Omega$  in spherical coordinates  $(\rho, \varphi, \theta)$  (see Example 3.3).
- (b) Compute  $d\Omega$  in both Cartesian and spherical coordinates and verify that both expressions represent the same 3-form.
- (c) Compute the restriction  $\Omega|_{\mathbb{S}^2} = \iota^* \Omega$ , using coordinates  $(\varphi, \theta)$ , on the open subset where these coordinates are defined.
- (d) Show that  $\Omega|_{\mathbb{S}^2}$  is nowhere zero.

## II. Optional problems.

4. Let V be a finite-dimensional vector space. We have two ways to think about the tensor space  $T^kV$ : concretely, as the space of k-multilinear functionals on V; and abstractly, as the tensor product space  $V^* \otimes \cdots \otimes V^*$ . However, we have defined alternating and symmetric tensors only in terms of the concrete definition. This problem outlines an abstract approach to alternating tensors.

Let  $\mathcal{A}$  denote the subspace of  $V^* \otimes \cdots \otimes V^*$  spanned by all elements of the form  $\alpha \otimes \xi \otimes \xi \otimes \beta$  for covectors  $\xi$  and arbitrary tensors  $\alpha, \beta$ , and let  $A^k V$  denote the quotient vector space  $V^* \otimes \cdots \otimes V^* / \mathcal{A}$ . Define a wedge product on  $A^k V$  by  $\omega \wedge \eta = \pi(\widetilde{\omega} \otimes \widetilde{\eta})$ , where  $\pi \colon V^* \otimes \cdots \otimes V^* \to A^k V$  is the projection, and  $\widetilde{\omega}, \widetilde{\eta}$  are arbitrary tensors such that  $\pi(\widetilde{\omega}) = \omega, \pi(\widetilde{\eta}) = \eta$ . Show that this wedge product is

well defined, and that there is a unique isomorphism  $F \colon \Lambda^k V \to A^k V$  such that the following diagram commutes:



and show that F takes the wedge product on  $\Lambda^k V$  (defined by the Alt convention) to the wedge product on  $A^k V$ . [This is another reason why the Alt convention for the wedge product is more natural than the determinant convention.]