Math 546 **Topology and Geometry of Manifolds** **Spring 2000**

ASSIGNMENT #8

DUE 5/19/2000

I. Required problems.

1. Let V be an n-dimensional vector space.
   (a) Show that covectors $\omega^1, \dots, \omega^k$ on V are linearly dependent if and only if $\omega^1 \wedge \cdots \wedge \omega^k = 0$.
   (b) Suppose $\{\omega^1, \dots, \omega^k\}$ and $\{\eta^1, \dots, \eta^k\}$ are two collections of independent covectors on V. Show that the collections have the same span if and only if for some nonzero constant c,
   $$\omega^1 \wedge \cdots \wedge \omega^k = c\, \eta^1 \wedge \cdots \wedge \eta^k.$$
   (c) If $\omega \in \Lambda^k V$, ω is said to be *decomposable* if ω can be written $\omega = \sigma^1 \wedge \cdots \wedge \sigma^k$, where each σ^i is a covector. Is every 2-covector on V decomposable? Your answer will depend on the dimension of V. Give proof or counterexample.
2. Prove Lemma 11.10.
3. Define a 2-form Ω on $\mathbb{R}^3$ by
   $$\Omega = x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy.$$
   (a) Compute Ω in spherical coordinates (ρ, φ, θ) (see Example 3.3).
   (b) Compute $d\Omega$ in both Cartesian and spherical coordinates and verify that both expressions represent the same 3-form.
   (c) Compute the restriction $\Omega|_{\mathbb{S}^2} = \iota^*\Omega$, using coordinates (φ, θ), on the open subset where these coordinates are defined.
   (d) Show that $\Omega|_{\mathbb{S}^2}$ is nowhere zero.

II. Optional problems.

4. Let V be a finite-dimensional vector space. We have two ways to think about the tensor space $T^k V$: concretely, as the space of k-multilinear functionals on V; and abstractly, as the tensor product space $V^* \otimes \cdots \otimes V^*$. However, we have defined alternating and symmetric tensors only in terms of the concrete definition. This problem outlines an abstract approach to alternating tensors.

   Let $\mathcal{A}$ denote the subspace of $V^* \otimes \cdots \otimes V^*$ spanned by all elements of the form $\alpha \otimes \xi \otimes \xi \otimes \beta$ for covectors ξ and arbitrary tensors α, β, and let $A^k V$ denote the quotient vector space $V^* \otimes \cdots \otimes V^* / \mathcal{A}$. Define a wedge product on $A^k V$ by $\omega \wedge \eta = \pi(\widetilde{\omega} \otimes \widetilde{\eta})$, where $\pi \colon V^* \otimes \cdots \otimes V^* \longrightarrow A^k V$ is the projection, and $\widetilde{\omega}$, $\widetilde{\eta}$ are arbitrary tensors such that $\pi(\widetilde{\omega}) = \omega$, $\pi(\widetilde{\eta}) = \eta$. Show that this wedge product is well defined, and that there is a unique isomorphism $F\colon \Lambda^k V \longrightarrow A^k V$ such that the following diagram commutes:

$$\begin{CD}
T^k V @>{\cong}>> V^* \otimes \cdots \otimes V^* \\
@V{\mathrm{Alt}}VV @VV{\pi}V \\
\Lambda^k V @>>{F}> A^k V,
\end{CD}$$

and show that F takes the wedge product on $\Lambda^k V$ (defined by the Alt convention) to the wedge product on $A^k V$. [This is another reason why the Alt convention for the wedge product is more natural than the determinant convention.]