Math 545 **Topology and Geometry of Manifolds** **Winter 2000**

Assignment #7

Due 2/25/2000

I. Required problems.

1. Prove the following refinement of Lemma 5.1: If $V: M \to TM$ is a (not necessarily continuous) map such that $V_p \in T_pM$ for each $p \in M$, then V is a smooth vector field if and only if Vf is a smooth function on M for every $f \in C^\infty(M)$.
2. If $N \subset M$ is a closed embedded submanifold and $V \in \mathcal{T}(N)$, show that there is a smooth vector field W on M such that $V = W|_N$.
3. Let E be a smooth rank-k vector bundle over a smooth manifold M, with projection $\pi: E \to M$, and let $U \subset M$ be an open set. A *local frame* for E over U is an ordered k-tuple $(\sigma_1, \dots, \sigma_k)$ where each σ_i is a smooth section of E over U (i.e., a smooth map $\sigma_i: U \to E$ such that $\pi \circ \sigma_i = \mathrm{Id}_U$), and such that $(\sigma_1|_p, \dots, \sigma_k|_p)$ is a basis for the fiber $\pi^{-1}(p)$ for each $p \in U$. It is called a *global frame* if $U = M$. Show that E admits a local frame over U if and only if it admits a local trivialization over U, and E admits a global frame if and only if it is trivial.
4. Find all integral curves of the following vector fields on the plane.
   (a) $V = 2x\dfrac{\partial}{\partial x} - y^2\dfrac{\partial}{\partial y}$.
   (b) $W = y\dfrac{\partial}{\partial x} - x\dfrac{\partial}{\partial y}$.

II. Optional problems.

5. A smooth manifold M is said to be *parallelizable* if its tangent bundle admits a global frame, which is equivalent by Problem 3 to TM being a trivial bundle. Show that the $\mathbb{T}^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ and $\mathbb{S}^3$ are parallelizable.