I. Required problems.

1. Prove the following refinement of Lemma 5.1: If $V : M \to TM$ is a (not necessarily continuous) map such that $V_p \in T_p M$ for each $p \in M$, then $V$ is a smooth vector field if and only if $V f$ is a smooth function on $M$ for every $f \in C^\infty(M)$.

2. If $N \subset M$ is a closed embedded submanifold and $V \in \mathcal{T}(N)$, show that there is a smooth vector field $W$ on $M$ such that $V = W|_N$.

3. Let $E$ be a smooth rank-$k$ vector bundle over a smooth manifold $M$, with projection $\pi : E \to M$, and let $U \subset M$ be an open set. A local frame for $E$ over $U$ is an ordered $k$-tuple $(\sigma_1, \ldots, \sigma_k)$ where each $\sigma_i$ is a smooth section of $E$ over $U$ (i.e., a smooth map $\sigma_i : U \to E$ such that $\pi \circ \sigma_i = \text{Id}_U$), and such that $(\sigma_1|_p, \ldots, \sigma_k|_p)$ is a basis for the fiber $\pi^{-1}(p)$ for each $p \in U$. It is called a global frame if $U = M$. Show that $E$ admits a local frame over $U$ if and only if it admits a local trivialization over $U$, and $E$ admits a global frame if and only if it is trivial.

4. Find all integral curves of the following vector fields on the plane.
   (a) $V = 2x \frac{\partial}{\partial x} - y^2 \frac{\partial}{\partial y}$.
   (b) $W = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}$.

II. Optional problems.

5. A smooth manifold $M$ is said to be parallelizable if its tangent bundle admits a global frame, which is equivalent by Problem 3 to $TM$ being a trivial bundle. Show that the $\mathbb{T}^n = S^1 \times \cdots \times S^1$ and $S^3$ are parallelizable.