Math 546 **Topology and Geometry of Manifolds** **Spring 2000**

Assignment #7

Due 5/12/2000

I. Required problems.

1. Let $\mathring{g} = \overline{g}|_{\mathbb{S}^n}$ denote the round metric on the n-sphere, i.e., the metric induced from the Euclidean metric by the usual inclusion of $\mathbb{S}^n$ into $\mathbb{R}^{n+1}$. Derive an expression for $\mathring{g}$ in stereographic coordinates by computing the pullback $(\sigma^{-1})^*\overline{g}$. Do the analogous computation in spherical coordinates $(x, y, z) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$.

2. Let (M, g) be an n-dimensional Riemannian manifold.

   (a) For any $p \in M$, show that there is an *orthonormal frame* on a neighborhood of p, that is, a smooth frame $(E_1, \dots, E_n)$ such that $g(E_i, E_j) = \delta_{ij}$. [Hint: start with a coordinate frame and use the Gram-Schmidt process. Be sure to show that the resulting frame is smooth.]

   (b) Observe that part (a) did *not* ask you to show that there are coordinates near p for which the *coordinate frame* is orthonormal. Show that there are such coordinates if and only if p has a neighborhood U such that (U, g) is isometric to an open subset of $\mathbb{R}^n$ with the Euclidean metric.

   (c) Now let $N \subset M$ be an embedded k-dimensional submanifold, endowed with the induced metric. Show that for any $p \in N$, there is an orthonormal frame $(E_1, \dots, E_n)$ for M on a neighborhood U of p in M such that, at each $p \in U \cap N$, $(E_1|_p, \dots, E_k|_p)$ is an orthonormal frame for T_pN. (Such a frame is called an *adapted orthonormal frame* for N.)

3. Let M be a compact Riemannian n-manifold, and $f \in C^\infty(M)$. Suppose f has only finitely many critical points $\{p_1, \dots, p_k\}$ with corresponding critical values $\{c_1, \dots, c_k\}$. (Assume without loss of generality that $c_1 \le \dots \le c_k$.) For any $a, b \in \mathbb{R}$, define $M_a = f^{-1}(a)$ and $M_{[a,b]} = f^{-1}([a, b])$. If a is a regular value, note that M_a is an embedded hypersurface in M.

   (a) Let X be the vector field $X = \operatorname{grad} f / |\operatorname{grad} f|^2$ on $M \smallsetminus \{p_1, \dots, p_k\}$, and let θ denote the flow of X. Show that $f(\theta_t(p)) = f(p) + t$ whenever $\theta_t(p)$ is defined.

   (b) Let $[a, b] \subset \mathbb{R}$ be an interval containing no critical values of f. Show that

   $$\theta : [0, b - a] \times M_a \longrightarrow M_{[a,b]}$$

   is a diffeomorphism, whose inverse is $p \mapsto \big(f(p) - a, \theta(a - f(p), p)\big)$.

   [Remark: This result shows that M can be decomposed as a union of simpler "building blocks"—the product manifolds $M_{[c_i+\varepsilon, c_{i+1}-\varepsilon]} \approx I \times M_{c_i+\varepsilon}$, and the neighborhoods $f^{-1}((c_i - \varepsilon, c_i + \varepsilon))$ of the critical points. This is the starting point of *Morse theory*, which is one of the deepest applications of differential geometry to topology. It is enlightening to think about what this means when M is a doughnut-shaped surface in $\mathbb{R}^3$ obtained by revolving a circle around the z-axis, and $f(x, y, z) = x$.]

II. Optional problems.

4. Let M be a smooth manifold. Show that TM is a trivial bundle if and only if T^*M is a trivial bundle.
5. Determine all Killing fields on $(\mathbb{R}^n, \overline{g})$.