Math 546 **Topology and Geometry of Manifolds** **Spring 2000**

ASSIGNMENT #6

DUE 5/4/2000

I. Required problems.

1. In each of the cases below, M is a smooth manifold and $f\colon M \longrightarrow \mathbb{R}$ is a smooth function. Compute the coordinate representation for df, and determine the set of all points $p \in M$ at which $df_p = 0$.
   (a) $M = \{(x, y) \in \mathbb{R}^2 : x > 0\}$; $f(x, y) = \tan^{-1}(y/x)$. Use standard coordinates (x, y).
   (b) M and f are as in part (1a); this time use polar coordinates (r, θ).
   (c) $M = S^2 \subset \mathbb{R}^3$; $f(p) = z(p)$ (the z-coordinate of p, thought of as a point in $\mathbb{R}^3$). Use stereographic coordinates.
   (d) $M = \mathbb{R}^n$; $f(x) = |x|^2$. Use standard coordinates.
2. Let $N \subset M$ be a connected immersed submanifold. Show that a function $f \in C^\infty(M)$ is constant on N if and only if $df|_N = 0$.
3. Let M be a smooth n-manifold, and σ a covariant k-tensor field on M. If $\{x^i\}$ and $\{\widetilde{x}^j\}$ are overlapping coordinate charts on M, we can write

   $$\sigma = \sigma_{i_1 \dots i_k} dx^{i_1} \otimes \cdots \otimes dx^{i_k} = \sigma = \widetilde{\sigma}_{j_1 \dots j_k} d\widetilde{x}^{j_1} \otimes \cdots \otimes d\widetilde{x}^{j_k}.$$

   Compute a transformation law analogous to (10.3) expressing the component functions $\sigma_{i_1 \dots i_k}$ in terms of $\widetilde{\sigma}_{j_1 \dots j_k}$.
4. Let V and W be finite-dimensional vector spaces. Prove that there is a natural (basis-independent) isomorphism between $V^* \otimes W$ and the space $\operatorname{Hom}(V, W)$ of linear maps from V to W.
5. Let M be a smooth manifold.
   (a) Show that a covariant k-tensor field τ is smooth if and only if, whenever $X_1, \dots, X_k$ are smooth vector fields defined on an open subset $U \subset M$, the function $\tau(X_1, \dots, X_k)$ defined in the obvious way is smooth on M.
   (b) Given a smooth k-tensor field τ, show that the map $\mathcal{T}(M) \times \cdots \times \mathcal{T}(M) \longrightarrow C^\infty(M)$ defined by

   $$(X_1, \dots, X_k) \mapsto \tau(X_1, \dots, X_k)$$

   is multilinear over $C^\infty(M)$, in the sense that for any smooth functions $f, f' \in C^\infty(M)$ and smooth vector fields X_i, X_i',

   $$\tau(X_1, \dots, fX_i + f'X_i', \dots, X_k) = f\tau(X_1, \dots, X_i, \dots, X_k) + f'\tau(X_1, \dots, X_i', \dots, X_k).$$

(c) Tensor Characterization Lemma: Show that a map

$$\widetilde{\tau}\colon \mathfrak{T}(M) \times \cdots \times \mathfrak{T}(M) \longrightarrow C^\infty(M)$$

is induced by a smooth tensor field as above if and only if it is multilinear over $C^\infty(M)$.

II. Optional problems.

6. Generalize problem 3 to mixed tensors of any rank.