I. Required problems.

1. Suppose $M$ is a smooth compact manifold.
   (a) If $f: M \to \mathbb{R}$ is a smooth function, show that $f$ vanishes at some point of $M$.
   (b) Show that there is no smooth submersion $F: M \to \mathbb{R}^k$ for any $k$.

2. Consider the map $F: \mathbb{R}^4 \to \mathbb{R}^2$ defined by
   $$F(x, y, s, t) = (x^2 + y, x^2 + y^2 + s^2 + t^2 + y).$$
   Show that $(0, 1)$ is a regular value of $F$, and that the level set $F^{-1}(0, 1)$ is diffeomorphic to $S^2$.

3. Exercise 3.4 in the notes.

4. Define subsets of $\mathbb{R}^2$ by
   $$M = \{(x, y) \in \mathbb{R}^2 : xy = 0\},$$
   $$N = \{(x, y) \in \mathbb{R}^2 : x^2 = y^3\}.$$ 
   Answer the following questions for each of these two subsets. Prove your answers correct.
   (a) Is it an embedded submanifold of $\mathbb{R}^2$?
   (b) If the answer to (a) is no, can it be given a smooth manifold structure (i.e., manifold topology and smooth structure) such that it is an immersed submanifold of $\mathbb{R}^2$?

II. Optional problems.

5. Let $F: M \to N$ be a smooth map of constant rank $k$, and let $S = F(M)$. Show that $S$ can be given a manifold topology and smooth structure such that it is an immersed $k$-dimensional submanifold of $N$ and $F: M \to S$ is smooth. Are these structures unique?

6. Decide whether each of the following statements is true or false, and discuss why.
   (a) If $F: M \to N$ is a smooth map and $F^{-1}(c)$ is an embedded submanifold of $M$ for some $c \in N$, then $c$ is a regular value of $F$.
   (b) If $S \subset M$ is a closed embedded submanifold, there is a smooth map $F: M \to P$ such that $S$ is a regular level set of $F$. 