Math 545 **Topology and Geometry of Manifolds** **Winter 2000**

Assignment #5

Due 2/9/2000

I. Required problems.

1. Suppose M is a smooth compact manifold.
   (a) If $f\colon M \longrightarrow \mathbb{R}$ is a smooth function, show that f_* vanishes at some point of M.
   (b) Show that there is no smooth submersion $F\colon M \longrightarrow \mathbb{R}^k$ for any k.
2. Consider the map $F\colon \mathbb{R}^4 \longrightarrow \mathbb{R}^2$ defined by
$$F(x, y, s, t) = (x^2 + y, x^2 + y^2 + s^2 + t^2 + y).$$
   Show that $(0, 1)$ is a regular value of F, and that the level set $F^{-1}(0, 1)$ is diffeomorphic to $\mathbb{S}^2$.
3. Exercise 3.4 in the notes.
4. Define subsets of $\mathbb{R}^2$ by
$$M = \{(x, y) \in \mathbb{R}^2 : xy = 0\},$$
$$N = \{(x, y) \in \mathbb{R}^2 : x^2 = y^3\}.$$
   Answer the following questions for each of these two subsets. Prove your answers correct.
   (a) Is it an embedded submanifold of $\mathbb{R}^2$?
   (b) If the answer to (a) is no, can it be given a smooth manifold structure (i.e., manifold topology and smooth structure) such that it is an immersed submanifold of $\mathbb{R}^2$?

II. Optional problems.

5. Let $F\colon M \longrightarrow N$ be a smooth map of constant rank k, and let $S = F(M)$. Show that S can be given a manifold topology and smooth structure such that it is an immersed k-dimensional submanifold of N and $F\colon M \longrightarrow S$ is smooth. Are these structures unique?
6. Decide whether each of the following statements is true or false, and discuss why.
   (a) If $F\colon M \longrightarrow N$ is a smooth map and $F^{-1}(c)$ is an embedded submanifold of M for some $c \in N$, then c is a regular value of F.
   (b) If $S \subset M$ is a closed embedded submanifold, there is a smooth map $F\colon M \longrightarrow P$ such that S is a regular level set of F.