Math 546 **Topology and Geometry of Manifolds** **Spring 2000**

Assignment #5

Due 4/28/2000

I. Required problems.

1. Let $A \subset \mathfrak{T}(\mathbb{R}^3)$ be the subspace with basis $\{X, Y, Z\}$, where

$$X = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \qquad Y = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}, \qquad Z = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}.$$

   Show that A is a Lie subalgebra of $\mathfrak{T}(\mathbb{R}^3)$, which is isomorphic (as a Lie algebra) to $\mathbb{R}^3$ with the cross product, and also to the Lie algebra $\mathfrak{o}(3)$ of $\mathrm{O}(3)$.

2. Let G be a connected Lie group and $\mathfrak{g}$ its Lie algebra.
   (a) If X, $Y \in \mathfrak{g}$, show that $[X, Y] = 0$ if and only if

$$\exp tX \exp sY = \exp sY \exp tX \text{ for all } s,\ t \in \mathbb{R}.$$

   (b) Show that G is abelian if and only if $\mathfrak{g}$ is abelian.
   (c) Give a counterexample when G is not connected.

3. Show that every connected abelian Lie group is Lie isomorphic to $\mathbb{R}^k \times \mathbb{T}^l$ for some nonnegative integers k and l.

II. Optional problems.

4. Define a map $\beta\colon \mathrm{GL}(n, \mathbb{C}) \longrightarrow \mathrm{GL}(2n, \mathbb{R})$ by identifying $(x^1 + iy^1, \ldots, x^n + iy^n) \in \mathbb{C}^n$ with $(x^1, y^1, \ldots, x^n, y^n) \in \mathbb{R}^{2n}$.
   (a) Show that β is an injective Lie group homomorphism, so that we can identify $\mathrm{GL}(n, \mathbb{C})$ with the Lie subgroup of $\mathrm{GL}(2n, \mathbb{R})$ consisting of matrices built up out of 2×2 blocks of the form $\left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right)$.
   (b) Under the usual identifications of $T_I \mathrm{GL}(n, \mathbb{C})$ with $M_{nn}(\mathbb{C})$ and $T_I \mathrm{GL}(2n, \mathbb{R})$ with $M_{2n,2n}(\mathbb{R})$, show that the induced homomorphism $\beta_*\colon \mathfrak{gl}(n, \mathbb{C}) \longrightarrow \mathfrak{gl}(2n, \mathbb{R})$ induces an injective Lie algebra homomorphism $M_{nn}(\mathbb{C}) \hookrightarrow M_{2n,2n}(\mathbb{R})$ (considering both as Lie algebras with the commutator bracket). Conclude that $\mathfrak{gl}(n, \mathbb{C})$ is isomorphic to the matrix algebra $M_{nn}(\mathbb{C})$.
   (c) Determine the Lie algebras of $\mathrm{SL}(n, \mathbb{C})$, $\mathrm{U}(n)$, and $\mathrm{SU}(n)$, as matrix subalgebras of $\mathfrak{gl}(n, \mathbb{C}) \cong M_{nn}(\mathbb{C})$.

5. (a) Let G and H be Lie groups. Suppose $\rho\colon H \times G \longrightarrow G$ is a smooth left action of H on G with the property that $\rho_h\colon G \longrightarrow G$ is a Lie group homomorphism for every $h \in H$. Define a group structure on the manifold $G \times H$ by

$$(g, h)(g', h') = (g\rho_h(g'), hh').$$

Show that this turns $G \times H$ into a Lie group, called the *semidirect product* of G and H induced by ρ, and denoted by $G \times_\rho H$.

(b) If G is any Lie group, show that G is Lie isomorphic to a semidirect product of a connected Lie group with a discrete group.

(c) Let $\mathfrak{g}$ be a finite-dimensional Lie algebra, and let G be the simply connected Lie group whose Lie algebra is $\mathfrak{g}$. Describe all Lie groups whose Lie algebra is $\mathfrak{g}$ in terms of G and discrete groups.