I. Required problems.

1. (a) Show that the image of a Lie group homomorphism is a Lie subgroup.
   (b) Show that the images of one-parameter subgroups in a Lie group $G$ are precisely the connected Lie subgroups of dimension less than or equal to 1.
   (c) If $H \subset G$ is the image of a one-parameter subgroup, show that $H$ is Lie isomorphic to one of the following: the trivial group $\{e\}$, $\mathbb{R}$, or $S^1$.

2. Prove that there is exactly one nonabelian 2-dimensional Lie algebra up to isomorphism.

3. Let $A$ and $B$ be the following elements of $\mathfrak{gl}(2, \mathbb{R})$:
   
   $$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. $$

   Compute the one-parameter subgroups of $\text{GL}(2, \mathbb{R})$ generated by $A$ and $B$.

4. (a) Suppose $A \in \text{GL}(n, \mathbb{R})$ is of the form $e^B$ for some $B \in \mathfrak{gl}(n, \mathbb{R})$. Show that $A$ has a square root, i.e., a matrix $C \in \text{GL}(n, \mathbb{R})$ such that $C^2 = A$.
   (b) Let
   
   $$A = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}. $$

   Show that the exponential map $\exp: \mathfrak{gl}(2, \mathbb{R}) \rightarrow \text{GL}(2, \mathbb{R})$ is not surjective, by showing that $A$ is not in its image.

II. Optional problems.

5. Let $\{i, j, k\}$ denote the standard basis of $\mathbb{R}^3$, and let $\mathbb{H} = \mathbb{R} \times \mathbb{R}^3$, with basis $\{1, i, j, k\}$. Define a bilinear multiplication $\mathbb{H} \times \mathbb{H} \rightarrow \mathbb{H}$ by setting
   
   $$1q = q1 = q \text{ for all } q \in \mathbb{H},$$
   $$ij = -ji = k,$$
   $$jk = -kj = i,$$
   $$ki = -ik = j,$$
   $$i^2 = j^2 = k^2 = -1,$$

   and extending bilinearly. With this multiplication, $\mathbb{H}$ is called the ring of quaternions.
   (a) Show that quaternionic multiplication is associative.
(b) Show that the set $S$ of unit quaternions (with respect to the Euclidean metric) is a Lie group under quaternionic multiplication, and is Lie isomorphic to SU(2).

(c) For any point $q \in \mathbb{H}$, show that the quaternions $iq$, $jq$, and $kq$ are orthogonal to $q$. Use this to define a left-invariant frame on $S$, and show that it corresponds under the isomorphism of (b) to the one defined in Example 9.6(c).

6. Look up the Cayley numbers, and use them to prove that $S^7$ is parallelizable by mimicking as much as you can of Problem 4. Why do the unit Cayley numbers not form a Lie group?