Math 546 **Topology and Geometry of Manifolds** **Spring 2000**

ASSIGNMENT #4

DUE 4/21/2000

I. Required problems.

1. (a) Show that the image of a Lie group homomorphism is a Lie subgroup.
   (b) Show that the images of one-parameter subgroups in a Lie group G are precisely the connected Lie subgroups of dimension less than or equal to 1.
   (c) If $H \subset G$ is the image of a one-parameter subgroup, show that H is Lie isomorphic to one of the following: the trivial group $\{e\}$, $\mathbb{R}$, or $\mathbb{S}^1$.
2. Prove that there is exactly one nonabelian 2-dimensional Lie algebra up to isomorphism.
3. Let A and B be the following elements of $\mathfrak{gl}(2,\mathbb{R})$:

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \qquad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

   Compute the one-parameter subgroups of $\mathrm{GL}(2,\mathbb{R})$ generated by A and B.
4. (a) Suppose $A \in \mathrm{GL}(n,\mathbb{R})$ is of the form e^B for some $B \in \mathfrak{gl}(n,\mathbb{R})$. Show that A has a square root, i.e., a matrix $C \in \mathrm{GL}(n,\mathbb{R})$ such that $C^2 = A$.
   (b) Let

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}.$$

   Show that the exponential map $\exp\colon \mathfrak{gl}(2,\mathbb{R}) \longrightarrow \mathrm{GL}(2,\mathbb{R})$ is not surjective, by showing that A is not in its image.

II. Optional problems.

5. Let $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$ denote the standard basis of $\mathbb{R}^3$, and let $\mathbb{H} = \mathbb{R} \times \mathbb{R}^3$, with basis $\{1,\mathbf{i},\mathbf{j},\mathbf{k}\}$. Define a bilinear multiplication $\mathbb{H} \times \mathbb{H} \longrightarrow \mathbb{H}$ by setting

$$\begin{aligned} &1q = q1 = q \text{ for all } q \in \mathbb{H}, \\ &\mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \\ &\mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \\ &\mathbf{ki} = -\mathbf{ik} = \mathbf{j}, \\ &\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \end{aligned}$$

   and extending bilinearly. With this multiplication, $\mathbb{H}$ is called the ring of *quaternions*.

   (a) Show that quaternionic multiplication is associative.

(b) Show that the set $\mathcal{S}$ of unit quaternions (with respect to the Euclidean metric) is a Lie group under quaternionic multiplication, and is Lie isomorphic to SU(2).

(c) For any point $q \in \mathbb{H}$, show that the quaternions $\mathbf{i}q$, $\mathbf{j}q$, and $\mathbf{k}q$ are orthogonal to q. Use this to define a left-invariant frame on $\mathcal{S}$, and show that it corresponds under the isomorphism of (b) to the one defined in Example 9.6(c).

6. Look up the Cayley numbers, and use them to prove that $\mathbb{S}^7$ is parallelizable by mimicking as much as you can of Problem 4. Why do the unit Cayley numbers not form a Lie group?