I. Required problems.

1. Let $V$ be a finite-dimensional vector space with its standard smooth structure. Show that for every $a \in V$ there is a natural (basis-independent) isomorphism $V \to T_a V$ such for any linear map $L: V \to W$ the following diagram commutes:

$$
\begin{array}{c}
V \\ \bigl\downarrow L \\
W \\
\end{array} \cong \begin{array}{c}
T_a V \\
L_* \\
T_{L(a)} W \\
\end{array}
$$

[Hint: look at the definition of the derivation $\bar{V}_a$ for a geometric tangent vector $V_a \in \mathbb{R}^n_a$.]

2. Let $M_{nn}(\mathbb{R})$ denote the $n^2$-dimensional vector space of $n \times n$ real matrices. For any $A \in M_{nn}(\mathbb{R})$, define

$$
e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k.
$$

(a) Show that the series converges uniformly on any bounded subset of $M_{nn}(\mathbb{R})$, and defines a smooth map $\exp: M_{nn}(\mathbb{R}) \to M_{nn}(\mathbb{R})$ defined by $\exp(A) = e^A$.

(b) Using the push-forward formula for tangent vectors to curves, compute $\exp_*: T_A M_{nn}(\mathbb{R}) \to T_{\exp A} M_{nn}(\mathbb{R})$. (Use the result of Problem 1 to identify both tangent spaces with $M_{nn}(\mathbb{R})$ itself.)

3. Let $M_1, \ldots, M_k$ be smooth manifolds. For any choices of points $p_i \in M_i$, $i = 1, \ldots, k$, show that the projection maps $\pi_j: M_1 \times \cdots \times M_k \to M_j$ yield an isomorphism

$$
\pi_1 \oplus \cdots \oplus \pi_k: T_{(p_1, \ldots, p_k)} (M_1 \times \cdots \times M_k) \to T_{p_1} M_1 \oplus \cdots \oplus T_{p_k} M_k.
$$

[Using this isomorphism, we will routinely identify $T_p M$ as a subspace of $T_{(p,q)} (M \times N)$.

4. Suppose $F: M \to N$ is a smooth map. Show that $F_*: TM \to TN$ is smooth.