Math 545 **Topology and Geometry of Manifolds** **Winter 2000**

Assignment #3

Due 1/26/2000

I. Required problems.

1. Let V be a finite-dimensional vector space with its standard smooth structure. Show that for every $a \in V$ there is a natural (basis-independent) isomorphism $V \longrightarrow T_aV$ such for any linear map $L\colon V \longrightarrow W$ the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{\cong} & T_aV \\ L\big\downarrow & & \big\downarrow L_* \\ W & \xrightarrow[\cong]{} & T_{L(a)}W \end{array}$$

[Hint: look at the definition of the derivation $\widetilde{V}_a$ for a geometric tangent vector $V_a \in \mathbb{R}^n_a$.]

2. Let $M_{nn}(\mathbb{R})$ denote the n^2-dimensional vector space of $n \times n$ real matrices. For any $A \in M_{nn}(\mathbb{R})$, define

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k.$$

   (a) Show that the series converges uniformly on any bounded subset of $M_{nn}(\mathbb{R})$, and defines a smooth map $\exp\colon M_{nn}(\mathbb{R}) \longrightarrow M_{nn}(\mathbb{R})$ defined by $\exp(A) = e^A$.

   (b) Using the push-forward formula for tangent vectors to curves, compute $\exp_*\colon T_AM_{nn}(\mathbb{R}) \longrightarrow T_{\exp A}M_{nn}(\mathbb{R})$. (Use the result of Problem 1 to identify both tangent spaces with $M_{nn}(\mathbb{R})$ itself.)

3. Let $M_1, \dots, M_k$ be smooth manifolds. For any choices of points $p_i \in M_i$, $i = 1, \dots, k$, show that the projection maps $\pi_j\colon M_1 \times \cdots \times M_k \longrightarrow M_j$ yield an isomorphism

$$\pi_1 \oplus \cdots \oplus \pi_k \colon T_{(p_1,\dots,p_k)}(M_1 \times \cdots \times M_k) \longrightarrow T_{p_1}M_1 \oplus \cdots \oplus T_{p_k}M_k.$$

[Using this isomorphism, we will routinely identify T_pM as a subspace of $T_{(p,q)}(M \times N)$.]

4. Suppose $F\colon M \longrightarrow N$ is a smooth map. Show that $F_*\colon TM \longrightarrow TN$ is smooth.