Math 546 **Topology and Geometry of Manifolds** **Spring 2000**

Assignment #3

Due 4/14/2000

I. Required problems.

1. Suppose G is a Lie group acting smoothly, freely, and properly on a smooth manifold M, and let N be any smooth manifold.
   (a) If $F\colon M/G \longrightarrow N$ is any map, show that F is smooth if and only if $F \circ \pi\colon M \longrightarrow N$ is smooth, where $\pi\colon M \longrightarrow M/G$ is the quotient map.
   (b) If $F\colon M \longrightarrow N$ is a smooth map, show that there is a smooth map $\widetilde{F}\colon M/G \longrightarrow N$ such that $\widetilde{F} \circ \pi = F$ if and only if F is G-invariant: $F(g \cdot p) = F(p)$ for all $g \in G$ and $p \in M$.
2. (a) Let G be a Lie group and H a closed normal Lie subgroup. Show that G/H is a Lie group and the quotient map $\pi\colon G \longrightarrow G/H$ is a Lie homomorphism.
   (b) If $F\colon G \longrightarrow H$ is a surjective Lie homomorphism, show that H is Lie isomorphic to $G/\operatorname{Ker}\theta$.
3. (a) Show that a surjective Lie homomorphism with discrete kernel is a smooth covering map.
   (b) Suppose G is a connected Lie group and $\pi\colon \widetilde{G} \longrightarrow G$ is any covering map. For any point $\widetilde{e} \in \pi^{-1}(e)$, show that $\widetilde{G}$ has a unique smooth structure and group structure such that $\widetilde{G}$ is a Lie group with identity $\widetilde{e}$ and π is a Lie homomorphism.
4. Prove that $\mathrm{SO}(3)$ is Lie isomorphic to $\mathrm{SU}(2)/\{\pm I\}$ and diffeomorphic to $\mathbb{P}^3$, as follows.
   (a) Let $\mathcal{H}$ denote the set of 2×2 trace-free Hermitian matrices. (A complex matrix A is *Hermitian* if $A = A^*$.) Show that $\mathcal{H}$ is a 3-dimensional vector space over $\mathbb{R}$, and

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

   is a basis for $\mathcal{H}$.
   (b) If we give $\mathcal{H}$ the Euclidean norm for which (e_1, e_2, e_3) is an orthonormal basis, show that $|A|^2 = -\det A$ for all $A \in \mathcal{H}$.
   (c) Identifying $\mathrm{GL}(3,\mathbb{R})$ with the set of invertible real-linear maps $\mathcal{H} \longrightarrow \mathcal{H}$ by means of the basis (e_1, e_2, e_3), define a map $\rho\colon \mathrm{SU}(2) \longrightarrow \mathrm{GL}(3,\mathbb{R})$ by

$$\rho(X)A = XAX^{-1}, \qquad X \in \mathrm{SU}(2),\ A \in \mathcal{H}.$$

   Show that ρ is a Lie group homomorphism whose image is $\mathrm{SO}(3)$ and whose kernel is $\{\pm I\}$. [Hint: to show the image is all of $\mathrm{SO}(3)$, show that ρ is open and closed and use the result of Problem 3 on Assignment 2. You may use without proof the fact that $\mathrm{SO}(3)$ is connected; see Problem 6.]
   (d) Prove the result.

II. Optional problems.

5. Determine which of the following Lie groups are compact: $\mathrm{GL}(n, \mathbb{R})$, $\mathrm{GL}(n, \mathbb{C})$, $\mathrm{SO}(n)$, $\mathrm{U}(n)$, $\mathrm{SU}(n)$, $\mathrm{O}(n)$.
6. Show that $\mathrm{SO}(n)$, $\mathrm{GL}(n, \mathbb{C})$, $\mathrm{U}(n)$, and $\mathrm{SU}(n)$ are connected, and that $\mathrm{GL}(n, \mathbb{R})$ and $\mathrm{O}(n)$ have exactly two components.
7. Prove that the group $E(n)$ acting on $\mathbb{R}^n$ as in Example 8.24(a) is precisely the set of Euclidean isometries of $\mathbb{R}^n$.
8. Prove that the Grassman manifold $G(k, n)$ is compact.