Math 545 **Topology and Geometry of Manifolds** **Winter 2000**

ASSIGNMENT #2

DUE 1/19/2000

I. Required problems.

1. Compute the coordinate representation for each of the following maps, using stereographic coordinates for spheres; use this to conclude that each map is smooth.
   (a) $\iota\colon \mathbb{S}^n \to \mathbb{R}^{n+1}$ is inclusion.
   (b) $A\colon \mathbb{S}^n \to \mathbb{S}^n$ is the antipodal map $A(x) = -x$.
   (c) $F\colon \mathbb{S}^3 \to \mathbb{S}^2$ is given by $F(z,w) = (z\overline{w} + w\overline{z}, iw\overline{z} - iz\overline{w}, z\overline{z} - w\overline{w})$, where we think of $\mathbb{S}^3$ as the subset $\{(w,z) : |w|^2 + |z|^2 = 1\}$ of $\mathbb{C}^2$.
2. Let $M = \overline{B}_1(0)$, the closed unit ball in $\mathbb{R}^n$. Show that M is a manifold with boundary, and can be given a smooth structure in such a way that the inclusion map $M \hookrightarrow \mathbb{R}^n$ is smooth.
3. Let M be a smooth n-manifold with boundary. Show that $\operatorname{Int} M$ is a smooth n-manifold and ∂M is a smooth $(n-1)$-manifold (both without boundary).

II. Optional problems.

4. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be the atlases for $\mathbb{R}$ defined by $\mathcal{A}_1 = \{(\mathbb{R}, \mathrm{Id})\}$, and $\mathcal{A}_2 = \{(\mathbb{R}, \psi)\}$, where $\psi(x) = x^3$. Let $f\colon \mathbb{R} \to \mathbb{R}$ be any function. Determine necessary and sufficient conditions on f so that it will be:
   (a) a smooth map $(\mathbb{R}, \mathcal{A}_2) \to (\mathbb{R}, \mathcal{A}_1)$;
   (b) a smooth map $(\mathbb{R}, \mathcal{A}_1) \to (\mathbb{R}, \mathcal{A}_2)$.
5. For any topological space X, let $C(X)$ denote the vector space of continuous functions $f\colon X \to \mathbb{R}$. If M and N are topological manifolds and $F\colon M \to N$ is a continuous map, define $F^*\colon C(N) \to C(M)$ by $F^*(f) = f \circ F$.
   (a) Show that F^* is linear.
   (b) Show that F is smooth if and only if $F^*(C^\infty(N)) \subset C^\infty(M)$.
   (c) If F is a homeomorphism, show that F is a diffeomorphism if and only if $F^*\colon C^\infty(N) \to C^\infty(M)$ is an isomorphism.

   Thus in a certain sense the entire smooth structure of M is encoded in the space $C^\infty(M)$.