Math 546 **Topology and Geometry of Manifolds** **Spring 2000**

ASSIGNMENT #2

DUE 4/7/2000

I. Required problems.

1. (a) Show that $\det\colon \mathrm{U}(n) \longrightarrow \mathbb{S}^1$ is a surjective Lie homomorphism.
   (b) Show that there exists a Lie homomorphism $\rho : \mathbb{S}^1 \longrightarrow \mathrm{U}(n)$ such that $\det \circ \rho = \mathrm{Id}_{\mathbb{S}^1}$.
   (c) Show that $\mathrm{U}(n)$ is diffeomorphic to $\mathbb{S}^1 \times \mathrm{SU}(n)$. Are they isomorphic Lie groups? [Hint: consider the map $\varphi : \mathbb{S}^1 \times \mathrm{SU}(n) \longrightarrow \mathrm{U}(n)$ given by $\varphi(z, A) = \rho(z)A$.]
2. Show that $\mathrm{SU}(2)$ is diffeomorphic to $\mathbb{S}^3$.
3. Let G be a Lie group, and let G_0 denote the connected component of the identity.
   (a) Show that G_0 is a Lie subgroup of G, and that each connected component of G is diffeomorphic to G_0.
   (b) If H is any connected open subgroup of G, show that $H = G_0$.
4. Suppose a Lie group G acts smoothly and freely on a smooth manifold M. Show that each orbit is an immersed submanifold of M.

II. Optional problems.

5. Prove the following partial converse to the quotient manifold theorem: if a Lie group G acts smoothly and freely on a smooth manifold M and the orbit space M/G has a smooth manifold structure such that the projection $\pi\colon M \longrightarrow M/G$ is a smooth submersion, then G acts properly.
6. Give an example of a smooth, proper action of a Lie group on a smooth manifold such that the orbit space is not a topological manifold.