Math 546 **Topology and Geometry of Manifolds** **Spring 2000**

Assignment #10

Due 6/2/2000

I. Required problems.

1. Let $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1 \subset \mathbb{R}^4$ denote the 2-torus, defined by $w^2 + x^2 = y^2 + z^2 = 1$. Compute $\int_{\mathbb{T}^2} \omega$, where ω is the following 2-form on $\mathbb{R}^4$:

$$\omega = wy\, dx \wedge dz.$$

2. Let ω be a 1-form on a smooth manifold M. Show that ω is conservative if and only if it is exact.

3. Let (M, g) be a compact, connected, oriented Riemannian manifold with boundary. Let $\widetilde{g}$ denote the induced metric on ∂M, and N the outward unit normal vector field along ∂M. The operator $\Delta\colon C^\infty(M) \longrightarrow C^\infty(M)$ defined by $\Delta u = \operatorname{div}(\operatorname{grad} u)$ is called the *Laplace operator*, and Δu is called the *Laplacian* of u. A function $u \in C^\infty(M)$ is said to be *harmonic* if $\Delta u = 0$.

   (a) Prove *Green's identities*:

$$\int_M u\Delta v\, dV_g + \int_M \langle \operatorname{grad} u, \operatorname{grad} v\rangle\, dV_g = \int_{\partial M} u\, Nv\, dV_{\widetilde{g}}. \tag{1}$$

$$\int_M (u\Delta v - v\Delta u)\, dV_g = \int_{\partial M} (u\, Nv - v\, Nu) dV_{\widetilde{g}}. \tag{2}$$

   (b) If $\partial M = \varnothing$, show that the only harmonic functions on M are the constants.

   (c) If $\partial M \neq \varnothing$, and u, v are harmonic functions on M whose restrictions to ∂M agree, show that $u \equiv v$.

4. Let T be a 2-tensor on a finite-dimensional vector space V. T is said to be *nondegenerate* if $T(X, Y) = 0$ for all Y implies $X = 0$. A *symplectic form* on V is a nondegenerate alternating 2-tensor. If T is a symplectic form on V, show that there exists a basis $(A_1, B_1, \dots, A_n, B_n)$ for V, with dual basis $(\alpha^1, \beta^1, \dots, \alpha^n, \beta^n)$, such that

$$T = \sum_{i=1}^{n} \alpha^i \wedge \beta^i.$$

   Conclude in particular that V is even-dimensional. [Hint: Show by induction that, for any k such that $0 \leq 2k \leq n$, there exist linearly independent vectors $\{A_1, B_1, \dots, A_k, B_k\}$ satisfying

$$T(A_i, A_j) = T(B_i, B_j) = 0;$$
$$T(A_i, B_j) = \tfrac{1}{2}\delta_{ij}.]$$

II. Optional problems.

5. Line Integrals of Vector Fields: Let (M, g) be a Riemannian manifold, and let $\gamma\colon [a, b] \longrightarrow M$ be an injective immersion.

   Let T denote the vector field on the image of γ defined by $T_{\gamma(t)} = \gamma'(t)/\left|\gamma'(t)\right|$. Let ds denote the Riemannian volume element on the image of γ with respect to the orientation determined by T.

   (a) For any smooth vector field X on M, show that

   $$\int_\gamma \langle X, T\rangle ds = \int_\gamma X^\flat.$$

   (b) Say X is conservative if and only if $\int_\gamma \langle X, T\rangle ds$ depends only on the endpoints of γ. Show that X is conservative if and only if it is the gradient of a smooth function.

6. Prove that the classical version of Stokes's theorem follows from the differential-forms version.