I. Required problems.

1. Suppose $V$ and $W$ are finite-dimensional vector spaces and $F: V \to W$ is any linear map. The rank of $F$ is the dimension of its image, and the nullity of $F$ is the dimension of its kernel.

   (a) Show that there are bases $\{E_1, \ldots, E_n\}$ for $V$ and $\{E'_1, \ldots, E'_m\}$ for $W$ with respect to which the matrix of $F$ has the block form
   
   $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$,

   where $I_r$ is the $r \times r$ identity matrix and $r$ is the rank of $F$.

   (b) Prove the rank-nullity law:

   $\text{rank } F + \text{nullity } F = \dim V$.

2. Let $F: \mathbb{R}^n \to \mathbb{R}^m$ and $G: \mathbb{R}^m \to \mathbb{R}^p$ be linear maps. Show that

   $|F(x)| \leq |F||x|,

   $|F \circ G| \leq |F||G|$,  

   where

   $|F| = \sqrt{\sum_{i,j} (F_i^j)^2}$  

   and $|G|$ is defined similarly. (Here $F_i^j$ are the matrix entries of $F$ with respect to the standard bases.)

3. Suppose $f_i: A \to \mathbb{R}$ is a sequence of real-valued continuous functions defined on a set $A \subset \mathbb{R}^n$.

   (a) Prove the Weierstrass $M$-test: if there exist positive real numbers $M_i$ such that $\sup_A |f_i| \leq M_i$ and $\sum_i M_i$ converges, then $\sum_i f_i$ converges uniformly on $A$.

   (b) If $f_i \to f$ uniformly on $A$, prove that $f$ is continuous.

   (c) If $A = [a_1, b_1] \times \cdots \times [a_n, b_n]$ is a closed $n$-dimensional rectangle and $f_i \to f$ uniformly on $A$, prove that

   $\lim_{i \to \infty} \int_A f_i \, dx^1 \cdots dx^n = \int_A f \, dx^1 \cdots dx^n$.  

(d) If $A$ is open, each $f_i$ is of class $C^1$, $f_i \to f$ pointwise on $A$, and $\partial f_i / \partial x^j \to g$ uniformly on $A$, prove that $\partial f / \partial x^j$ exists and

$$\frac{\partial f}{\partial x^j} = \lim_{i \to \infty} \frac{\partial f_i}{\partial x^j}.$$ 

4. Let $f: U \to \mathbb{R}$ be a smooth function on a convex open set $U \subset \mathbb{R}^n$. Prove Taylor’s formula with remainder: for any $a \in U$,

$$f(x) = f(a) + \frac{\partial f}{\partial x^i}(a)(x^i - a^i) + g_i(x)(x^i - a^i),$$

(Using the summation convention) where $g_i: U \to \mathbb{R}$ are smooth functions that vanish at $a$. [Hint: apply the fundamental theorem of calculus, the chain rule, and Problem 5 to

$$\int_0^1 \frac{\partial}{\partial t} f(a + t(x - a)) \, dt.$$]

II. Optional problems.

5. Let $U \subset \mathbb{R}^n$ be an open set, $a, b \in \mathbb{R}$, and let $f: U \times [a, b] \to \mathbb{R}$ be a smooth function. Define $F: U \to \mathbb{R}$ by

$$F(x) = \int_a^b f(x, t) \, dt.$$ 

Show that $F$ is smooth, and its derivatives can be computed by differentiating under the integral sign:

$$\frac{\partial F}{\partial x^i}(x) = \int_a^b \frac{\partial f}{\partial x^i}(x, t) \, dt.$$