Math 545 **Topology and Geometry of Manifolds** **Winter 2000**

Assignment #1

Due 1/12/2000

I. Required problems.

1. Suppose V and W are finite-dimensional vector spaces and $F\colon V \to W$ is any linear map. The *rank* of F is the dimension of its image, and the *nullity* of F is the dimension of its kernel.
   (a) Show that there are bases $\{E_1, \dots, E_n\}$ for V and $\{E'_1, \dots, E'_m\}$ for W with respect to which the matrix of F has the block form
   $$\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix},$$
   where I_r is the $r \times r$ identity matrix and r is the rank of F.
   (b) Prove the *rank-nullity law*:
   $$\operatorname{rank} F + \operatorname{nullity} F = \dim V.$$
2. Let $F\colon \mathbf{R}^n \to \mathbf{R}^m$ and $G\colon \mathbf{R}^m \to \mathbf{R}^p$ be linear maps. Show that
   $$|F(x)| \le |F|\,|x|,$$
   $$|F \circ G| \le |F|\,|G|,$$
   where
   $$|F| = \sqrt{\textstyle\sum_{i,j} (F_i^j)^2}$$
   and $|G|$ is defined similarly. (Here F_i^j are the matrix entries of F with respect to the standard bases.)
3. Suppose $f_i\colon A \to \mathbf{R}$ is a sequence of real-valued continuous functions defined on a set $A \subset \mathbf{R}^n$.
   (a) Prove the *Weierstrass M-test:* if there exist positive real numbers M_i such that $\sup_A |f_i| \le M_i$ and $\sum_i M_i$ converges, then $\sum_i f_i$ converges uniformly on A.
   (b) If $f_i \to f$ uniformly on A, prove that f is continuous.
   (c) If $A = [a_1, b_1] \times \cdots \times [a_n \times b_n]$ is a closed n-dimensional rectangle and $f_i \to f$ uniformly on A, prove that
   $$\lim_{i \to \infty} \int_A f_i \, dx^1 \cdots dx^n = \int_A f \, dx^1 \cdots dx^n.$$

(d) If A is open, each f_i is of class C^1, $f_i \to f$ pointwise on A, and $\partial f_i/\partial x^j \to g$ uniformly on A, prove that $\partial f/\partial x^j$ exists and

$$\frac{\partial f}{\partial x^j} = \lim_{i\to\infty} \frac{\partial f_i}{\partial x^j}.$$

4. Let $f\colon U \to \mathbf{R}$ be a smooth function on a convex open set $U \subset \mathbf{R}^n$. Prove *Taylor's formula with remainder:* for any $a \in U$,

$$f(x) = f(a) + \frac{\partial f}{\partial x^i}(a)(x^i - a^i) + g_i(x)(x^i - a^i),$$

(using the summation convention) where $g_i\colon U \to \mathbf{R}$ are smooth functions that vanish at a. [Hint: apply the fundamental theorem of calculus, the chain rule, and Problem 5 to

$$\int_0^1 \frac{\partial}{\partial t} f(a + t(x - a))\, dt.]$$

II. Optional problems.

5. Let $U \subset \mathbf{R}^n$ be an open set, $a, b \in \mathbf{R}$, and let $f\colon U \times [a, b] \to \mathbf{R}$ be a smooth function. Define $F\colon U \to \mathbf{R}$ by

$$F(x) = \int_a^b f(x, t)\, dt.$$

Show that F is smooth, and its derivatives can be computed by differentiating under the integral sign:

$$\frac{\partial F}{\partial x^i}(x) = \int_a^b \frac{\partial f}{\partial x^i}(x, t)\, dt.$$