Math 544: Hint for Problem 5-1

This problem turned out to be unreasonably hard as stated, and will not be counted as part of the regular homework assignment, but you may turn it in for extra credit if you wish. Here are some suggestions for one way to go about it.

First do the noncompact case. Choose an arbitrary point $q_0 \in M$, and show that there is a countable open cover of M, say $\{U_1, U_2, \ldots\}$, with the following properties: (1) U_i is homeomorphic to an open interval; (2) $\overline{U_i}$ is homeomorphic to a closed interval; (3) $U_i \subset \overline{U_i} \subset U_{i+1}$ for each i. Now show that there exists a homeomorphism $\phi\colon \mathbb{R} \to M$ sending $\{\pm i\}$ to ∂U_i.

Now if M is compact and p_0 is any point in M, by the preceding argument $U := M - \{p_0\}$ is homeomorphic to R. Choose a neighborhood V of q_0 such that $\overline{V}$ is homeomorphic to $[0,1]$, and show there exists a quotient map $\pi\colon [0,1] \cup [2,3] \to M$ that makes the identifications $1 \leftrightarrow 2$ and $0 \leftrightarrow 3$. Conclude that M is homeomorphic to $\mathbb{S}^1$.