Math 544/5/6 **Topology and Geometry of Manifolds** **2006–2007**

Assignment #7, Supplementary Problems

Required Problem

A. Suppose K and L are Euclidean simplicial complexes and $f\colon |K| \to |L|$ is a simplicial map with vertex map $f_0\colon K^{(0)} \to L^{(0)}$.

(a) Show that f is an embedding if and only if f_0 is injective.

(b) Show that f is a quotient map if and only if f_0 is surjective.

B. Define abstract simplicial complexes whose geometric realizations are homeomorphic to the following spaces:

(a) $[0, \infty)$.

(b) $\mathbb{R}^2$.

(c) The wedge sum of n circles.

(d) The wedge sum of two copies of $\S^2$.

Optional Problem

C. This problem describes a way to attach a "geometric realization" to any abstract simplicial complex, not necessarily satisfying the hypotheses of Problem 5-4. Let $\mathscr{K}$ be an abstract simplicial complex and let $S = \mathscr{K}_0$ be its set of vertices. Define $\mathbb{R}\langle S\rangle$ to be subset of $\prod_{v\in S}\mathbb{R}$ consisting of those elements $(x_v)_{v\in S}$ such that $x_v = 0$ for all but finitely many v. It is a vector space with the obvious operations of scalar multiplication and vector addition: $a(x_v)_{v\in S} + b(y_v)_{v\in S} = (ax_v + by_v)_{v\in S}$. There is a natural injective map of S into $\mathbb{R}\langle S\rangle$ defined by sending v to the element with $x_v = 1$ and $x_{v'} = 0$ for $v' \neq v$. If we identify S with its image under this injection, then each point $x \in \mathbb{R}\langle S\rangle$ can be written uniquely as a finite linear combination $\sum_{v\in S} x_v v$, with $x_v \in \mathbb{R}$ and $x_v = 0$ for all but finitely many v. Define a metric $d(x,y) = \left(\sum_{v\in S}(x_v - y_v)^2\right)^{1/2}$, and note that any finite-dimensional linear subspace is isometric to $\mathbb{R}^n$ in the obvious way.

For each $\sigma = \{v_0, \dots, v_k\} \in \mathscr{K}$, define the ***geometric realization*** of σ to be the set

$$|\sigma| = \left\{ \sum_{i=0}^{k} t_i v_i : 0 \le t_i \le 1, \text{ and } \sum_i t_i = 1 \right\} \subseteq \mathbb{R}\langle S\rangle,$$

with the metric topology. Then we define the geometric realization of $\mathscr{K}$ to be the set

$$|\mathscr{K}| = \bigcup_{\sigma\in\mathscr{K}} |\sigma|,$$

and define a topology on $|\mathscr{K}|$ (not the metric topology in general) to be the topology coherent with the geometric realizations of its simplices.

(a) For each vertex $v \in \mathscr{K}_0$, define a function $t_v\colon |\mathscr{K}| \to \mathbb{R}$ by letting $t_v(x)$ be the coefficient of v in the linear combination defining x. Show that t_v is continuous.

(b) Use part (a) to show that $|\mathscr{K}|$ is Hausdorff.

(c) For each vertex $v \in \mathscr{K}_0$, let $\operatorname{St} v$ (the ***open star*** of v) be the union of the interiors of all simplices having v as a vertex. Show that $\operatorname{St} v$ is a neighborhood of v, and the collection of all open stars is an open cover of $|\mathscr{K}|$.

(d) Show that any compact subset of $|\mathscr{K}|$ is contained in the union of finitely many open simplices.

(e) Show that $|\mathscr{K}|$ is locally compact if and only if $\mathscr{K}$ is locally finite.

(f) Show that $|\mathscr{K}|$ is second countable if and only if $\mathscr{K}$ is countable and locally finite.

(g) If K is a Euclidean geometric complex and $\mathscr{K}$ is its vertex scheme, show that $|\mathscr{K}|$ is homeomorphic to $|K|$.