The Postulates of Neutral Geometry

Axiom 1 (The Set Postulate). *Every line is a set of points, and the collection of all points forms a set $\mathbb{P}$ called **the plane**.*

Axiom 2 (The Existence Postulate). *There exist at least two distinct points.*

Axiom 3 (The Incidence Postulate). *For every pair of distinct points P and Q, there exists exactly one line ℓ such that both P and Q lie on ℓ.*

Axiom 4 (The Distance Postulate). *For every pair of points P and Q, the **distance from P to Q**, denoted by PQ, is a nonnegative real number determined uniquely by P and Q.*

Axiom 5 (The Ruler Postulate). *For every line ℓ, there is a bijective function $f\colon \ell \to \mathbb{R}$ with the property that for any two points $P, Q \in \ell$, we have*

$$PQ = |f(Q) - f(P)|.$$

*Any function with these properties is called a **coordinate function for ℓ**.*

Axiom 6 (The Plane Separation Postulate). *If ℓ is a line, the **sides of ℓ** are two disjoint, nonempty sets of points whose union is the set of all points not on ℓ. If P and Q are distinct points not on ℓ, then both of the following equivalent conditions are satisfied:*

(*i*) *P and Q are on the same side of ℓ if and only if $\overline{PQ} \cap \ell = \varnothing$.*

(*ii*) *P and Q are on opposite sides of ℓ if and only if $\overline{PQ} \cap \ell \neq \varnothing$.*

Axiom 7 (The Angle Measure Postulate). *For every angle $\angle ABC$, the **measure of $\angle ABC$**, denoted by $\mu\angle ABC$, is a real number strictly between 0 and 180, determined uniquely by $\angle ABC$.*

Axiom 8 (The Protractor Postulate). *For every half-rotation $\mathrm{HR}(A, O, B)$, there is a bijective function $g\colon \mathrm{HR}(A, O, B) \to [0, 180] \subseteq \mathbb{R}$, which assigns the number 0 to $\overrightarrow{OA}$ and the number 180 to the ray opposite $\overrightarrow{OA}$, and such that if $\overrightarrow{OC}$ and $\overrightarrow{OD}$ are any two noncollinear rays in $\mathrm{HR}(A, O, B)$, then*

$$\mu\angle COD = \left| g\big(\overrightarrow{OD}\big) - g\big(\overrightarrow{OC}\big) \right|.$$

*Any such function is called a **coordinate function for $\mathrm{HR}(A, O, B)$**.*

Axiom 9 (The SAS Postulate). *If there is a correspondence between the vertices of two triangles such that two sides and the included angle of one triangle are congruent to the corresponding sides and angle of the other triangle, then the triangles are congruent.*

Theorems of Neutral Geometry

Theorem 1.1. *If ℓ and m are distinct, nonparallel lines, then there exists a unique point P such that P lies on both ℓ and m.*

Corollary 1.2 (Trichotomy for Lines). *If ℓ and m are two lines, then exactly one of the following conditions holds: Either $\ell = m$, or $\ell \parallel m$, or ℓ intersects m at exactly one point.*

Theorem 1.3. *If P and Q are points (distinct or not), then there is a line that contains them.*

Theorem 1.6 (Ruler Placement Theorem). *Suppose ℓ is a line and P, Q are two distinct points on ℓ. Then there exists a coordinate function $f\colon \ell \to \mathbb{R}$ such that $f(P) = 0$ and $f(Q) > 0$.*

Theorem 1.7 (Properties of Distances). *If P and Q are any two points, their distance has the following properties:*

(*a*) *$PQ = QP$.*

(*b*) *$PQ > 0$ if and only if $P \neq Q$.*

(*c*) *$PQ = 0$ if and only if $P = Q$.*

Theorem 1.8. *Every line contains infinitely many distinct points.*

Corollary 1.9. *There exist infinitely many distinct points.*

Theorem 1.11 (Symmetry of Betweenness of Points). *If A, B, C are any three points, then $A * B * C$ if and only if $C * B * A$.*

Theorem 1.12 (Betweenness Theorem for Points). *If $A * B * C$, then $AB + BC = AC$.*

Theorem 1.13. *If A, B, and C are three distinct collinear points, then exactly one of them lies between the other two.*

Corollary 1.14 (Consistency of Betweenness of Points). *If A, B, C are three points on a line ℓ and $A * B * C$, then $f(B)$ is between $f(A)$ and $f(C)$ for every coordinate function $f\colon \ell \to \mathbb{R}$.*

Theorem 1.15 (Partial Converse to the Betweenness Theorem for Points). *If A, B, C are three distinct collinear points such that $AB + BC = AC$, then $A * B * C$.*

Proposition 1.16. *Suppose A, B, C, D are four distinct points. If any of the following pairs of conditions holds, then $A * B * C * D$:*

$$A * B * C \text{ and } B * C * D; \qquad \text{or} \qquad A * B * C \text{ and } A * C * D; \qquad \text{or} \qquad A * B * D \text{ and } B * C * D.$$

Conversely, if $A * B * C * D$, then all four of the following are true:

$$A * B * C, \qquad A * B * D, \qquad A * C * D, \qquad B * C * D.$$

Proposition 2.1 (Euclid's Common Notions for Segments).

- (a) **Transitivity of Congruence:** *Two segments that are both congruent to a third segment are congruent to each other.*
- (b) **Segment Addition Theorem:** *Suppose $A * B * C$ and $A' * B' * C'$. If $\overline{AB} \cong \overline{A'B'}$ and $\overline{BC} \cong \overline{B'C'}$, then $\overline{AC} \cong \overline{A'C'}$.*
- (c) **Segment Subtraction Theorem:** *Suppose $A*B*C$ and $A'*B'*C'$. If $\overline{AC} \cong \overline{A'C'}$ and $\overline{AB} \cong \overline{A'B'}$, then $\overline{BC} \cong \overline{B'C'}$.*
- (d) **Reflexivity of Congruence:** *Every segment is congruent to itself.*
- (e) **The Whole Segment is Greater Than the Part:** *If $A * B * C$, then $AC > AB$ and $AC > BC$.*

Lemma 2.2 (Coordinate Representation of a Segment). *Suppose A and B are distinct points, and $f\colon \overleftrightarrow{AB} \to \mathbb{R}$ is a coordinate function for $\overleftrightarrow{AB}$. Then*

$$\overline{AB} = \{P \in \overleftrightarrow{AB} : f(A) \le f(P) \le f(B)\} \qquad \text{if } f(A) < f(B);$$
$$\overline{AB} = \{P \in \overleftrightarrow{AB} : f(A) \ge f(P) \ge f(B)\} \qquad \text{if } f(A) > f(B).$$

Proposition 2.3. *If $A * B * C$, then the following set equalities hold:*

- (a) $\overline{AB} \cup \overline{BC} = \overline{AC}$.
- (b) $\overline{AB} \cap \overline{BC} = \{B\}$.

Lemma 2.4. *Let $\overline{AB}$ be a segment and let M be a point. If either of the following conditions holds, then M is the midpoint of $\overline{AB}$.*

- (a) $M \in \overline{AB}$ *and* $AM = \frac{1}{2}AB$.
- (b) $M \in \overleftrightarrow{AB}$ *and* $AM = MB$.

Theorem 2.5 (Existence and Uniqueness of Midpoints). *Every segment has a unique midpoint.*

Proposition 2.6. *Every segment contains infinitely many distinct points.*

Lemma 2.7 (Coordinate Representation of a Ray). *Suppose A and B are distinct points, and $f\colon \overleftrightarrow{AB} \to \mathbb{R}$ is a coordinate function for $\overleftrightarrow{AB}$. Then*

$$\overrightarrow{AB} = \{P \in \overleftrightarrow{AB} : f(P) \ge f(A)\} \qquad \text{if } f(A) < f(B);$$
$$\overrightarrow{AB} = \{P \in \overleftrightarrow{AB} : f(P) \le f(A)\} \qquad \text{if } f(A) > f(B).$$

Theorem 2.8. *If $\overrightarrow{AD}$ is a ray and B and C are interior points of $\overrightarrow{AD}$ such that $AC > AB$, then $A * B * C$.*

Proposition 2.9. *Suppose A and B are distinct points, and P is a point on the line $\overleftrightarrow{AB}$. Then $P \notin \overrightarrow{AB}$ if and only if $P * A * B$.*

Theorem 2.10 (Segment Construction Theorem). *Suppose $\overrightarrow{AB}$ is a ray and r is a positive real number. Then there exists a unique point $C \in \overrightarrow{AB}$ such that $AC = r$.*

Corollary 2.11. *If $\overline{AB}$ and $\overline{CD}$ are segments with $AB > CD$, there is a point E in the interior of $\overline{AB}$ such that $\overline{AE} \cong \overline{CD}$.*

Theorem 2.12 (Segment Extension Theorem). *If $\overline{AB}$ is any segment, there exist points $C, D \in \overleftrightarrow{AB}$ such that $C * A * B$ and $A * B * D$.*

Theorem 2.13 (Properties of Rays with the Same Endpoint). *Suppose $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are rays with the same endpoint.*

(a) *If $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are collinear, then they are either equal or opposite, but not both.*

(b) *If $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are opposite rays, then $\overrightarrow{AB} \cap \overrightarrow{AC} = \{A\}$ and $\overrightarrow{AB} \cup \overrightarrow{AC} = \overleftrightarrow{AC}$.*

(c) *$\overrightarrow{AB}$ and $\overrightarrow{AC}$ are the same ray if and only if they have an interior point in common.*

(d) *$\overrightarrow{AB}$ and $\overrightarrow{AC}$ are opposite rays if and only if $C * A * B$.*

Proposition 2.14. *There exist three noncollinear points.*

Theorem 2.15. *Every half-plane is a convex set.*

Theorem 2.16 (The Y-Theorem). *Suppose ℓ is a line, A is a point on ℓ, and B is a point not on ℓ. Then every interior point of $\overrightarrow{AB}$ is on the same side of ℓ as B.*

Theorem 2.17 (The X-Theorem). *Suppose $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are opposite rays, and ℓ is a line that intersects $\overleftrightarrow{AB}$ only at O. Then $\overrightarrow{OA}$ and $\overrightarrow{OB}$ lie on opposite sides of ℓ.*

Exercise 2.7. *Suppose A and B are two distinct points.*

(a) $\overrightarrow{AB} \cap \overrightarrow{BA} = \overline{AB}$.

(b) $\overrightarrow{AB} \cup \overrightarrow{BA} = \overleftrightarrow{AB}$.

Proposition 3.1. *All right angles are congruent.*

Theorem 3.2 (Angle Construction Theorem). *Let $\overrightarrow{OA}$ be a ray and let S be a side of $\overleftrightarrow{OA}$. For every real number m such that $0 < m < 180$, there is a unique ray $\overrightarrow{OC}$ starting at O and lying on side S such that $\mu\angle AOC = m^\circ$.*

Proposition 3.3. *Supplements of congruent angles are congruent, and complements of congruent angles are congruent.*

Theorem 3.4 (Linear Pair Theorem). *If two angles form a linear pair, they are supplementary.*

Corollary 3.5. *If two angles in a linear pair are congruent, then they are both right angles.*

Theorem 3.6 (Vertical Angles Theorem). *Vertical angles are congruent.*

Theorem 3.7 (Symmetry of Betweenness of Rays). *If $\overrightarrow{OA}$, $\overrightarrow{OB}$, and $\overrightarrow{OC}$ are rays with a common endpoint, then $\overrightarrow{OA} * \overrightarrow{OB} * \overrightarrow{OC}$ if and only if $\overrightarrow{OC} * \overrightarrow{OB} * \overrightarrow{OA}$.*

Theorem 3.8 (Betweenness Theorem for Rays). *If $\overrightarrow{OA} * \overrightarrow{OB} * \overrightarrow{OC}$, then $\mu\angle AOB + \mu\angle BOC = \mu\angle AOC$.*

Proposition 3.9 (Euclid's Common Notions for Angles).

(a) **Transitivity of Congruence:** *Two angles that are both congruent to a third angle are congruent to each other.*

(b) **Angle Addition Theorem:** *Suppose $\overrightarrow{OA} * \overrightarrow{OB} * \overrightarrow{OC}$ and $\overrightarrow{O'A'} * \overrightarrow{O'B'} * \overrightarrow{O'C'}$. If $\angle AOB \cong \angle A'O'B'$ and $\angle BOC \cong \angle B'O'C'$, then $\angle AOC \cong \angle A'O'C'$.*

(c) **Angle Subtraction Theorem:** *Suppose $\overrightarrow{OA} * \overrightarrow{OB} * \overrightarrow{OC}$ and $\overrightarrow{O'A'} * \overrightarrow{O'B'} * \overrightarrow{O'C'}$. If $\angle AOC \cong \angle A'O'C'$ and $\angle AOB \cong \angle A'O'B'$, then $\angle BOC \cong \angle B'O'C'$.*

(d) **Reflexivity of Congruence:** *Every angle is congruent to itself.*

(e) **The Whole Angle is Greater Than the Part:** *If $\overrightarrow{OA} * \overrightarrow{OB} * \overrightarrow{OC}$, then $\mu\angle AOC > \mu\angle AOB$.*

Theorem 3.10. *If $\overrightarrow{OA}$, $\overrightarrow{OB}$, and $\overrightarrow{OC}$ are rays with a common endpoint, no two of which are collinear, and all lying in a single half-rotation, then exactly one of them lies between the other two.*

Corollary 3.11 (Consistency of Betweenness of Rays). *Suppose $\overrightarrow{OA} * \overrightarrow{OB} * \overrightarrow{OC}$. If $\mathrm{HR}(D, O, E)$ is any half-rotation containing all three rays and g is a corresponding coordinate function, then $g(\overrightarrow{OB})$ is between $g(\overrightarrow{OA})$ and $g(\overrightarrow{OC})$.*

Theorem 3.12 (Linear Triple Theorem). *If $\angle AOB$, $\angle BOC$, and $\angle COD$ form a linear triple, then their measures add up to 180°.*

Theorem 3.13 (Existence and Uniqueness of Angle Bisectors). *Every angle has a unique angle bisector.*

Proposition 3.14. *If $\overrightarrow{OA} * \overrightarrow{OB} * \overrightarrow{OC}$, then A and C lie on opposite sides of $\overleftrightarrow{OB}$.*

Theorem 3.16 (The 360 Theorem). *Suppose $\overrightarrow{OA}$, $\overrightarrow{OB}$, and $\overrightarrow{OC}$ are three distinct rays with the same endpoint, such that no two of the rays are collinear and none of the rays lies in the interior of the angle formed by the other two. Then*

$$\mu\angle AOB + \mu\angle BOC + \mu\angle AOC = 360^\circ.$$

Theorem 3.17 (Betweenness vs. Interior). *Suppose $\overrightarrow{OA}$, $\overrightarrow{OB}$, and $\overrightarrow{OC}$ are three rays with the same endpoint such that no two are collinear. Then $\overrightarrow{OB}$ lies in the interior of $\angle AOC$ if and only if $\overrightarrow{OA} * \overrightarrow{OB} * \overrightarrow{OC}$.*

Theorem 3.18 (Betweenness vs. Betweenness). *Suppose ℓ is a line, O is a point not on ℓ, and A, B, C are three distinct points on ℓ. Then $A * B * C$ if and only if $\overrightarrow{OA} * \overrightarrow{OB} * \overrightarrow{OC}$.*

Exercise 3.2 (Converse to the Linear Pair Theorem). *Suppose $\overrightarrow{OA}$, $\overrightarrow{OB}$, and $\overrightarrow{OC}$ are three rays with the same endpoint, such that A and C are on opposite sides of $\overleftrightarrow{OB}$, and $\mu\angle AOB + \mu\angle BOC = 180°$. Then $\overrightarrow{OA}$ and $\overrightarrow{OC}$ are opposite rays.*

Exercise 3.9. *Suppose $\overrightarrow{OA}$ is a ray and B and C are points on the same side of $\overleftrightarrow{OA}$. If $\mu\angle AOB < \mu\angle AOC$, then $\overrightarrow{OB}$ is between $\overrightarrow{OA}$ and $\overrightarrow{OC}$.*

Exercise 3.10 (Converse to the Betweenness Theorem for Rays). If $\overrightarrow{OA}$, $\overrightarrow{OB}$, and $\overrightarrow{OC}$ are three rays with the same endpoint such that no two are collinear and $\mu\angle AOB + \mu\angle BOC = \mu\angle AOC$, then $\overrightarrow{OA} * \overrightarrow{OB} * \overrightarrow{OC}$.

Exercise 3.11.

(*a*) *If $S_1, \dots, S_k$ are convex subsets of the plane, then $S_1 \cap \cdots \cap S_k$ is convex.*

(*b*) *The interior of an angle is a convex set.*

Theorem 4.1 (Pasch's Theorem). *Suppose $\triangle ABC$ is a triangle and ℓ is a line that does not contain any of the points A, B, or C. If ℓ intersects one of the sides of $\triangle ABC$, then it also intersects another side.*

Theorem 4.2 (The Crossbar Theorem). *If $\triangle ABC$ is a triangle and $\overrightarrow{AD}$ is a ray between $\overrightarrow{AB}$ and $\overrightarrow{AC}$, then $\overrightarrow{AD}$ intersects the interior of $\overline{BC}$.*

Proposition 4.3 (Transitivity of Congruence of Triangles). *Two triangles that are both congruent to a third triangle are congruent to each other.*

Theorem 4.4 (ASA). *If there is a correspondence between the vertices of two triangles such that two angles and the included side of one triangle are congruent to the corresponding angles and side of the other triangle, then the triangles are congruent.*

Theorem 4.5 (Isosceles Triangle Theorem). *If two sides of a triangle are congruent to each other, then the angles opposite those sides are congruent.*

Theorem 4.6 (Converse to the Isosceles Triangle Theorem). *If two angles of a triangle are congruent to each other, then the sides opposite those angles are congruent.*

Corollary 4.7. *A triangle is equilateral if and only if it is equiangular.*

Theorem 4.8 (Copying a Triangle). *Suppose $\triangle ABC$ is a triangle, and $\overline{DE}$ is a segment congruent to $\overline{AB}$. On each side of $\overleftrightarrow{DE}$, there is a unique point F such that $\triangle DEF \cong \triangle ABC$.*

Theorem 4.9 (SSS). *If there is a correspondence between the vertices of two triangles such that all three sides of one triangle are congruent to the corresponding sides of the other triangle, then the triangles are congruent.*

Theorem 4.10 (Exterior Angle Theorem). *The measure of an exterior angle is strictly greater than the measure of either remote interior angle.*

Corollary 4.11. *In any triangle, the two smallest angles are acute.*

Theorem 4.12 (Scalene Inequality). *Let $\triangle ABC$ be a triangle. Then $AC > BC$ if and only if $\mu\angle B > \mu\angle A$.*

Corollary 4.13. *In any right triangle, the hypotenuse is strictly longer than either leg.*

Theorem 4.14 (Triangle Inequality). *If A, B, and C are noncollinear points, then $AB + BC > AC$.*

Corollary 4.15 (Converse to the Betweenness Theorem for Points). *If A, B, and C are three distinct points and $AB + BC = AC$, then $A * B * C$.*

Theorem 4.16 (AAS). *If there is a correspondence between the vertices of two triangles such that two angles and a non-included side of one triangle are congruent to the corresponding angles and side of the other triangle, then the triangles are congruent.*

Theorem 4.17 (HL). *If the hypotenuse and one leg of a right triangle are congruent to the hypotenuse and one leg of another, then the triangles are congruent.*

Exercise 4.1. *Suppose $\triangle ABC$ is a triangle and ℓ is a line. If ℓ intersects $\triangle ABC$, then it intersects at least two of its sides.*

Exercise 4.5. *Suppose $\triangle ABC$ and $\triangle DEF$ are two triangles such that $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, and $\angle A \cong \angle D$ (the hypotheses of ASS). Then $\angle C$ and $\angle F$ are either congruent or supplementary.*

Theorem 6.1 (Four Right Angles Theorem). *If $\ell \perp m$, then ℓ and m form four right angles.*

Theorem 6.2 (Constructing a Perpendicular). *Let ℓ be a line and let P be a point on ℓ. Then there exists a unique line m that is perpendicular to ℓ at P.*

Theorem 6.3 (Dropping a Perpendicular). *Suppose ℓ is a line and P is a point not on ℓ. Then there exists a unique line m that contains P and is perpendicular to ℓ.*

Theorem 6.4. *If $\triangle ABC$ is a triangle, then the altitude to $\overline{AB}$ intersects the interior of $\overline{AB}$ if and only if $\angle A$ and $\angle B$ are both acute.*

Corollary 6.5. *In any triangle, the altitude to the longest side always intersects the interior of that side.*

Corollary 6.6. *In a right triangle, the altitude to the hypotenuse always intersects the interior of the hypotenuse.*

Theorem 6.7 (Existence and Uniqueness of a Perpendicular Bisector). *Every segment has a unique perpendicular bisector.*

Theorem 6.8 (Perpendicular Bisector Theorem). *Suppose $\overline{AB}$ is a segment. A point P lies on the perpendicular bisector of $\overline{AB}$ if and only if it is equidistant from A and B.*

Theorem 6.9 (Shortest Distance from a Point to a Line). *Suppose ℓ is a line and P is a point not on ℓ. Let F be the foot of the perpendicular from P to ℓ. Then the distance from P to F is strictly smaller than the distance from P to any other point on ℓ.*

Theorem 6.10 (Angle Bisector Theorem). *Suppose $\angle AOB$ is an angle and P is a point in the interior of $\angle AOB$. Then P lies on the bisector of $\angle AOB$ if and only if it is equidistant from $\overleftrightarrow{OA}$ and $\overleftrightarrow{OB}$.*

Theorem 6.11 (Alternate Interior Angles Theorem). *If two lines are cut by a transversal making a pair of congruent alternate interior angles, then they are parallel.*

Corollary 6.12 (Corresponding Angles Theorem). *If two lines are cut by a transversal making a pair of congruent corresponding angles, then they are parallel.*

Corollary 6.13 (Consecutive Interior Angles Theorem). *If two lines are cut by a transversal making a pair of supplementary consecutive interior angles, then they are parallel.*

Corollary 6.14 (Common Perpendiculars Theorem). *If two lines have a common perpendicular (i.e., a line that is perpendicular to both), then they are parallel.*

Theorem 6.15 (Existence of Parallels). *For every line ℓ and every point P that does not lie on ℓ, there exists a line m such that P lies on m and $m \parallel \ell$. It can be chosen so that ℓ and m have a common perpendicular that contains P.*

Theorem 7.1. *A quadrilateral is convex if and only if each vertex lies in the interior of the angle at the opposite vertex.*

Lemma 7.2. *A quadrilateral is convex if and only if both pairs of opposite sides are semiparallel.*

Theorem 7.3. *If a quadrilateral has at least one pair of semiparallel sides, it is convex.*

Corollary 7.4. *Every trapezoid is a convex quadrilateral.*

Corollary 7.5. *Every parallelogram is a convex quadrilateral.*

Theorem 7.6 (Truncated Triangle Theorem). *Suppose $\triangle ABC$ is a triangle, and D and E are points such that $A * D * C$ and $A * E * B$. Then $BCDE$ is a convex quadrilateral.*

Theorem 7.7. *The diagonals of a convex quadrilateral intersect at a point that is in the interior of both diagonals.*

Theorem 7.8 (SASAS). *Suppose $ABCD$ and $EFGH$ are convex quadrilaterals such that $\overline{AB} \cong \overline{EF}$, $\overline{BC} \cong \overline{FG}$, $\overline{CD} \cong \overline{GH}$, $\angle B \cong \angle F$, and $\angle C \cong \angle G$. Then $ABCD \cong EFGH$.*

Theorem 7.9 (AASAS). *Suppose $ABCD$ and $EFGH$ are convex quadrilaterals such that $\angle A \cong \angle E$, $\angle B \cong \angle F$, $\angle C \cong \angle G$, $\overline{BC} \cong \overline{FG}$, and $\overline{CD} \cong \overline{GH}$. Then $ABCD \cong EFGH$.*

Theorem 7.10 (Copying a Quadrilateral). *Suppose $ABCD$ is a convex quadrilateral, and $\overline{EF}$ is a segment congruent to $\overline{AB}$. On either side of $\overleftrightarrow{EF}$, there are points G and H such that $EFGH \cong ABCD$.*

Theorem 9.1 (Polygon Decomposition Theorem). *Suppose $\mathscr{P}$ is a convex polygon, and B and C are any two points on $\mathscr{P}$ that do not lie on any one edge of $\mathscr{P}$. Then the segment $\overline{BC}$ divides $\mathscr{P}$ into two convex polygons that form an admissible decomposition of $\mathscr{P}$.*

Lemma 13.3 (Hinge Theorem). *Suppose $\triangle ABC$ and $\triangle DEF$ are two triangles such that $\overline{AB} \cong \overline{DE}$ and $\overline{BC} \cong \overline{EF}$. Then $\mu\angle B > \mu\angle E$ if and only if $AC > DF$.*

Theorem 13.4 (Saccheri–Legendre). *In neutral geometry, the angle sum of every triangle is less than or equal to 180°.*

Corollary 13.5 (Saccheri–Legendre Theorem for Convex Polygons). *In neutral geometry, if $\mathscr{P}$ is a convex n-sided polygon, then the angle sum of $\mathscr{P}$ satisfies*

$$\sigma(\mathscr{P}) \le \frac{n-2}{2} \times 180^\circ.$$

Theorem 13.6 (Additivity of Defects). *In neutral geometry, suppose $\mathscr{P}$ is a convex polygon, and B and C are two points on $\mathscr{P}$ that are not contained in any one edge of $\mathscr{P}$. Let $\mathscr{P}_1$ and $\mathscr{P}_2$ be the two convex polygons formed by $\overline{BC}$ as in the polygon decomposition theorem (Theorem 9.1). Then $\delta(\mathscr{P}) = \delta(\mathscr{P}_1) + \delta(\mathscr{P}_2)$.*

Theorem 13.10 (The All-or-Nothing Theorem). *In neutral geometry, if there exist one line ℓ_0 and one point $P_0 \notin \ell_0$ such that there are two or more distinct lines parallel to ℓ_0 through P_0, then for every line ℓ and every point $P \notin \ell$, there are two or more distinct lines parallel to ℓ through P.*

Corollary 13.11. *In every model of neutral geometry, either the Euclidean parallel postulate or the hyperbolic parallel postulate holds.*

Lemma 15.1. *Suppose $\overrightarrow{OA}$, $\overrightarrow{OB}$, and $\overrightarrow{OC}$ are three rays that all lie in some half-rotation. If A and C are on opposite sides of $\overleftrightarrow{OB}$, then $\overrightarrow{OA} * \overrightarrow{OB} * \overrightarrow{OC}$.*

Lemma 15.2. *Suppose $\overrightarrow{OA}$, $\overrightarrow{OB}$, and $\overrightarrow{OC}$ are rays such that $\overrightarrow{OA} * \overrightarrow{OB} * \overrightarrow{OC}$. Then*

$$\operatorname{Int}\angle AOC = \operatorname{Int}\angle AOB \cup \operatorname{Int}\overrightarrow{OB} \cup \operatorname{Int}\angle BOC. \tag{15.1}$$

Lemma 15.3 (Small Angle Lemma). *Suppose $\overrightarrow{PQ}$ is a ray, and A is a point not on $\overleftrightarrow{PQ}$. Given any positive number ε, there exists a point X in the interior of $\overrightarrow{PQ}$ such that $\mu\angle AXP < \varepsilon$.*

The Postulates of Euclidean Geometry

Axioms 1–9 of neutral geometry.

Axiom 10 (The Euclidean Parallel Postulate). *For every line ℓ and for every point P that does not lie on ℓ, there is exactly one line m such that P lies on m and $m \parallel \ell$.*

Axiom 11 (The Area Postulate). *Associated with every polygonal region R, there is a positive real number $\alpha(R)$ called the* ***area of R****, which satisfies the following conditions:*

(*i*) (Congruence Property of Area) *If R_1 and R_2 are congruent triangular regions, then*

$$\alpha(R_1) = \alpha(R_2).$$

(*ii*) (Additivity Property of Area) *If $R_1, \dots, R_n$ are nonoverlapping polygonal regions, then*

$$\alpha(R_1 \cup \cdots \cup R_n) = \alpha(R_1) + \cdots + \alpha(R_n).$$

Axiom 12 (The Unit Area Postulate). *The area of a unit square is 1.*

Theorems of Euclidean Geometry

All of the theorems of neutral geometry.

Theorem 6.16 (Converse to the Alternate Interior Angles Theorem). *If two parallel lines are cut by a transversal, then both pairs of alternate interior angles are congruent.*

Corollary 6.17 (Converse to the Corresponding Angles Theorem). *If two parallel lines are cut by a transversal, then all four pairs of corresponding angles are congruent.*

Corollary 6.18 (Converse to the Consecutive Interior Angles Theorem). *If two parallel lines are cut by a transversal, then each pair of consecutive interior angles is supplementary.*

Theorem 6.19 (Parallel Lines are Equidistant Lines). *If ℓ and m are distinct lines, then ℓ is equidistant from m if and only if it is parallel to m.*

Corollary 6.20 (Symmetry of Equidistant Lines). *If ℓ and m are two distinct lines, then ℓ is equidistant from m if and only if m is equidistant from ℓ.*

Lemma 6.21 (Proclus's Lemma). *If ℓ and ℓ' are parallel lines and t is a line that is distinct from ℓ but intersects ℓ, then t also intersects ℓ'.*

Theorem 6.22. *If ℓ and ℓ' are parallel lines, then any line perpendicular to one of them is perpendicular to both.*

Corollary 6.23 (Converse to the Common Perpendiculars Theorem). *If two lines are parallel, then they have a common perpendicular.*

Theorem 6.24 (Transitivity of Parallelism). *If ℓ, m, and n are distinct lines such that $\ell \parallel m$ and $m \parallel n$, then $\ell \parallel n$.*

Theorem 6.25 (Angle-Sum Theorem). *Every triangle has angle sum equal to $180°$.*

Corollary 6.26. *In any triangle, the sum of the measures of any two interior angles is less than $180°$.*

Corollary 6.27. *In any triangle, the measure of each exterior angle is equal to the sum of the measures of the two remote interior angles.*

Theorem 6.28 (The 60-60-60 Theorem). *A triangle is equilateral if and only if all three of its interior angles measure $60°$.*

Theorem 6.29 (The 30-60-90 Theorem). *A triangle has interior angle measures $30°$, $60°$, and $90°$ if and only if it is a right triangle in which the length of the hypotenuse is twice the length of the shortest leg.*

Theorem 6.30 (The 45-45-90 Theorem). *A triangle has interior angle measures $45°$, $45°$, and $90°$ if and only if it is an isosceles right triangle.*

Theorem 6.31 (The Euclidean Parallel Postulate Implies Euclid's Postulate 5). *If ℓ and ℓ' are two lines cut by a transversal t in such a way that the measures of two consecutive interior angles add up to less than $180°$, then ℓ and ℓ' intersect on the side of t where those two angles lie.*

Theorem 7.11 (Angle-Sum Theorem for Convex Quadrilaterals). *If $ABCD$ is a convex quadrilateral, then $\sigma(ABCD) = 360°$.*

Theorem 7.12. *Every parallelogram has the following properties.*

- (a) *Both pairs of opposite sides are congruent.*
- (b) *Both pairs of opposite angles are congruent.*
- (c) *Its diagonals bisect each other.*

Theorem 7.13. *Every rectangle has the following properties.*

- (a) *It is a parallelogram.*
- (b) *Its diagonals are congruent.*

Theorem 7.14. *Every rhombus has the following properties.*

- (a) *It is a parallelogram.*
- (b) *Its diagonals intersect perpendicularly.*

Exercise 7.2. *If the diagonals of a quadrilateral intersect, then the quadrilateral is convex.*

Exercise 7.6. *Suppose $ABCD$ is a convex quadrilateral, and E and F are points such that $A * E * B$ and $D * F * C$. Then $AEFD$ and $BEFC$ are convex quadrilaterals.*

Exercise 7.11. *Suppose $ABCD$ is a quadrilateral. Then each of the following conditions is sufficient to conclude that $ABCD$ is a parallelogram.*

- (a) *The diagonals bisect each other.*
- (b) *Both pairs of opposite sides are congruent.*
- (c) *A pair of opposite sides are parallel and congruent.*

Exercise 7.12. *Suppose $ABCD$ is a quadrilateral.*

- (a) *If its diagonals are congruent and bisect each other, then $ABCD$ is a rectangle.*

(*b*) *If its diagonals are perpendicular and bisect each other, then $ABCD$ is a rhombus.*

Theorem 8.1. *A polygon is convex if and only if each vertex lies in the interior of each nonadjacent angle.*

Theorem 8.2. *A polygon is convex if and only if all pairs of nonadjacent edges are semiparallel.*

Theorem 8.3. *Every closed half-plane is a convex set.*

Theorem 8.4. *A polygon $\mathscr{P}$ is convex if and only if for each edge of $\mathscr{P}$, the entire polygon $\mathscr{P}$ is contained in one of the closed half-planes determined by that edge.*

Corollary 8.5. *If $\mathscr{P}$ is a convex polygon and $\overline{AB}$ is an edge of $\mathscr{P}$, then $\mathscr{P} \cap \overleftrightarrow{AB} = \overline{AB}$.* □

Theorem 8.6 (Angle-Sum Theorem for Convex Polygons). *In a convex polygon with n sides, the angle sum is equal to $(n-2) \times 180°$.*

Corollary 8.7. *In a regular polygon with n sides, the measure of each angle is $\frac{n-2}{n} \times 180°$.* □

Lemma 8.8. *Suppose $\mathscr{P}$ is a polygon and Q is a point not on $\mathscr{P}$. Then all rays starting at Q and not containing any vertices of $\mathscr{P}$ have the same parity.*

Theorem 8.9. *Let $\mathscr{P}$ be a polygon, and let X be a point on $\mathscr{P}$.*

(*a*) *If X lies on an edge $\overline{AB}$ but is not a vertex, then every ray starting at X and lying on one side of $\overleftrightarrow{AB}$ is inward-pointing, while every ray on the other side is outward-pointing.*

(*b*) *If X is a vertex of $\mathscr{P}$, then either every ray in the interior of $\angle X$ is inward-pointing and every ray in the exterior of $\angle X$ is outward-pointing, or vice versa.*

Lemma 8.10. *Every polygon has at least one convex vertex.*

Lemma 8.11. *Every polygon has at least one interior diagonal.*

Theorem 8.12 (Angle-Sum Theorem for General Polygons). *If $\mathscr{P}$ is any polygon with n sides, the sum of its interior angle measures is $(n-2) \times 180°$.*

Theorem 8.13. *Suppose $\mathscr{P}$ is a convex polygon. For any point Q in the plane, the following are equivalent:*

(*a*) *Q is in the interior of $\mathscr{P}$;*
(*b*) *Q is in the interior of each of the angles of $\mathscr{P}$;*
(*c*) *Q is on the same side of each edge of $\mathscr{P}$ as the vertices not on that edge.*

Theorem 8.14. *Suppose $\mathscr{P}$ is a convex polygon. A point Q is in $\mathrm{Reg}(\mathscr{P})$ if and only if it lies in every closed half-plane determined by an edge of $\mathscr{P}$ and the vertices not on that edge.*

Theorem 8.15 (Characterizations of Convex Polygons). *If $\mathscr{P}$ is a polygon, the following are equivalent:*

(*a*) *$\mathscr{P}$ is a convex polygon;*
(*b*) *$\mathrm{Int}(\mathscr{P})$ is a convex set;*
(*c*) *$\mathrm{Reg}(\mathscr{P})$ is a convex set;*
(*d*) *All vertices of $\mathscr{P}$ are convex.*

Theorem A (from the blog). *If ℓ and m are two distinct lines and there are two distinct points on ℓ that are on the same side of m and equidistant from m, then ℓ is parallel to m.*

Lemma 9.2 (Rectangle Decomposition Lemma). *Suppose $ABCD$ is a rectangle, and E, F, G, H are interior points on $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$, respectively, such that $\overleftrightarrow{EG} \parallel \overleftrightarrow{AD}$ and $\overleftrightarrow{HF} \parallel \overleftrightarrow{AB}$. Then there is a point X where $\overline{HF}$ intersects $\overline{EG}$, and*

$$\alpha(ABCD) = \alpha(AEXH) + \alpha(EBFX) + \alpha(HXGD) + \alpha(XFCG).$$

Theorem 9.6 (Area of a Square). *The area of a square of side x is x^2.*

Theorem 9.7 (Area of a Rectangle). *The area of a rectangle is the product of the lengths of any two adjacent sides.*

Lemma 9.8 (Area of a Right Triangle). *The area of a right triangle is one-half of the product of the lengths of its legs.*

Theorem 9.9 (Area of a Triangle). *The area of a triangle is equal to one-half the length of any base multiplied by the corresponding height.*

Corollary 9.10 (Triangle Sliding Lemma). *Suppose $\triangle ABC$ and $\triangle A'BC$ are two distinct triangles that have a common side $\overline{BC}$, such that $\overleftrightarrow{AA'} \parallel \overleftrightarrow{BC}$. Then $\alpha(\triangle ABC) = \alpha(\triangle A'BC)$.*

Corollary 9.11. *Suppose $\triangle ABC$ is a triangle, and D is a point such that $B * D * C$. Then*

$$\frac{\alpha(\triangle ABD)}{\alpha(\triangle ABC)} = \frac{BD}{BC}.$$

Theorem 9.12 (Area of a Parallelogram). *The area of a parallelogram is the product of the length of any base times the corresponding height.*

Theorem 9.13 (Area of a Trapezoid). *The area of a trapezoid is the product of the height times the average of the lengths of the bases.*

Theorem 10.1 (The Side-Splitter Theorem). *Suppose $\triangle ABC$ is a triangle, and ℓ is a line parallel to $\overleftrightarrow{BC}$ that intersects $\overline{AB}$ at an interior point D. Then ℓ also intersects $\overline{AC}$ at an interior point E, and the following proportions hold:*

$$\frac{AD}{AB} = \frac{AE}{AC} \qquad \text{and} \qquad \frac{AD}{DB} = \frac{AE}{EC}.$$

Theorem 10.2 (AA Similarity Theorem). *If there is a correspondence between the vertices of two triangles such that two pairs of corresponding angles are congruent, then the triangles are similar.*

Theorem 10.3 (SAS Similarity Theorem). *If $\triangle ABC$ and $\triangle DEF$ are triangles such that $\angle A \cong \angle D$ and $AB/DE = AC/DF$, then $\triangle ABC \sim \triangle DEF$.*

Theorem 10.4 (SSS Similarity Theorem). *If $\triangle ABC$ and $\triangle DEF$ are triangles such that $AB/DE = AC/DF = BC/EF$, then $\triangle ABC \sim \triangle DEF$.*

Theorem 10.5 (Converse to the Side-Splitter Theorem). *Suppose $\triangle ABC$ is a triangle, and D and E are interior points on $\overline{AB}$ and $\overline{AC}$, respectively, such that*

$$\frac{AD}{AB} = \frac{AE}{AC}.$$

Then $\overleftrightarrow{DE}$ is parallel to $\overleftrightarrow{BC}$.

Theorem 10.6 (Similar Triangle Construction Theorem). *If $\triangle ABC$ is a triangle and $\overline{DE}$ is any segment, then on each side of $\overleftrightarrow{DE}$, there is a unique point F such that $\triangle ABC \sim \triangle DEF$.*

Theorem 10.7 (Angle Bisector Proportionality Theorem). *Suppose $\triangle ABC$ is a triangle and D is a point on $\overline{BC}$ such that $\overrightarrow{AD}$ is the bisector of $\angle BAC$. Then*

$$\frac{BD}{DC} = \frac{AB}{AC}.$$

Theorem 10.8 (Parallel Projection Theorem). *Suppose ℓ, m, and n are three distinct parallel lines, and t and t' are two other lines such that t intersects ℓ, m, and n at A, B, and C, respectively, and t' intersects them at A', B', and C', respectively. If B is between A and C, then*

$$\frac{AB}{BC} = \frac{A'B'}{B'C'}.$$

Theorem 10.9 (Perimeter Scaling Theorem). *If two polygons are similar, then the ratio of their perimeters is the same as the ratio of their corresponding side lengths.*

Theorem 10.10 (Height Scaling Theorem). *If two triangles are similar, their corresponding heights have the same ratio as their corresponding side lengths.*

Theorem 10.11 (Diagonal Scaling Theorem). *If two convex quadrilaterals are similar, the lengths of their corresponding diagonals have the same ratio as their corresponding side lengths.*

Theorem 10.12 (Triangle Area Scaling Theorem). *If two triangles are similar, then the ratio of their areas is the square of the ratio of their corresponding side lengths; that is, if $\triangle ABC \sim \triangle DEF$ and $AB = r \cdot DE$, then*

$$\alpha(\triangle ABC) = r^2 \cdot \alpha(\triangle DEF).$$

Theorem 10.13 (Quadrilateral Area Scaling Theorem). *If two convex quadrilaterals are similar, then the ratio of their areas is the square of the ratio of their corresponding side lengths.*

Theorem 11.1 (The Pythagorean Theorem). *Suppose $\triangle ABC$ is a right triangle with right angle at C, and let a, b, and c denote the lengths of the sides opposite A, B, and C, respectively. Then $a^2 + b^2 = c^2$.*

Lemma 11.2 (Lemma B from the Blog). *Suppose ℓ, m, and n are three distinct parallel lines, and t and t' are two other lines such that t intersects ℓ, m, and n at A, B, and C, respectively, and t' intersects them at A', B', and C', respectively. If B is between A and C, then B' is between A' and C'.*

Lemma 11.3 (Constructing a Square). *Suppose $\overline{AB}$ is a segment. On either side of $\overleftrightarrow{AB}$, there are points C and D such that $ABCD$ is a square.*

Theorem 11.4 (Converse to the Pythagorean Theorem). *Suppose $\triangle ABC$ is a triangle with side lengths a, b, and c. If $a^2 + b^2 = c^2$, then $\triangle ABC$ is a right triangle, and its hypotenuse is the side of length c.*

Theorem 11.5 (Side Lengths of 30-60-90 Triangles). *In a triangle with angle measures $30°$, $60°$, and $90°$, the longer leg is $\sqrt{3}$ times as long as the shorter leg, and the hypotenuse is twice as long as the shorter leg.*

Theorem 11.6 (Side Lengths of Isosceles Right Triangles). *In an isosceles right triangle, the hypotenuse is $\sqrt{2}$ times as long as either leg.*

Theorem 11.7 (Diagonal of a Square). *In a square, each diagonal is $\sqrt{2}$ times as long as each side.*

Theorem 11.8 (SSS Triangle Construction Theorem). *Suppose a, b, and c are positive real numbers such that each one is strictly less than the sum of the other two. Then there exists a triangle with side lengths a, b, and c.*

Corollary 11.9. *Suppose a, b, and c are positive real numbers such that each one is strictly less than the sum of the other two, and $\overline{AB}$ is a segment of length c. Then on either side of $\overleftrightarrow{AB}$, there exists a unique point C such that $AC = b$ and $BC = a$.*

Theorem 11.10 (Right Triangle Similarity Theorem). *The altitude to the hypotenuse of a right triangle forms two triangles that are similar to each other and to the original triangle.*

Theorem 11.11 (Right Triangle Proportion Theorem). *In every right triangle, the following proportions hold:*

(*a*) *The altitude to the hypotenuse is the geometric mean between the projections of the two legs.*

(*b*) *Each leg is the geometric mean between its projection and the hypotenuse.*

Theorem 12.1. *If $\mathscr{C}$ is a circle of radius r, then the length of every diameter of $\mathscr{C}$ is $2r$.*

Theorem 12.2. *Two concentric circles that have a point in common are equal.*

Theorem 12.3 (Chord Theorem). *Suppose $\mathscr{C}$ is a circle and $\overline{AB}$ is a chord of $\mathscr{C}$.*

(*a*) *The perpendicular bisector of $\overline{AB}$ passes through the center of $\mathscr{C}$.*

(*b*) *A radius of $\mathscr{C}$ is perpendicular to $\overline{AB}$ if and only if it bisects $\overline{AB}$.*

Theorem 12.4. *No circle contains three distinct collinear points.*

Theorem 12.5 (Tangent Line Theorem). *Suppose $\mathscr{C}$ is a circle, and ℓ is a line that intersects $\mathscr{C}$ at a point P. Then ℓ is tangent to $\mathscr{C}$ if and only if ℓ is perpendicular to the radius through P.*

Theorem 12.6. *If $\mathscr{C}$ is a circle and ℓ is a line that is tangent to $\mathscr{C}$ at P, then every point of ℓ except P lies in the exterior of $\mathscr{C}$.*

Theorem 12.7 (Line-Circle Theorem). *Suppose $\mathscr{C}$ is a circle and ℓ is a line that contains a point in the interior of $\mathscr{C}$. Then ℓ is a secant line for $\mathscr{C}$.*

Theorem 12.8. *Suppose $\mathscr{C}$ is a circle and ℓ is a secant line that intersects $\mathscr{C}$ at A and B. Then every interior point of the chord $\overline{AB}$ is in the interior of $\mathscr{C}$, and every point of ℓ that is not in $\overline{AB}$ is in the exterior of $\mathscr{C}$.*

Lemma 12.9 (Polygon Inequality). *Suppose n is an integer greater than or equal to 3, and $A_1, \dots, A_n$ are any n points, not necessarily all distinct. Then*

$$A_1A_n \leq A_1A_2 + A_2A_3 + \dots A_{n-1}A_n.$$

Theorem 12.10 (The Two-Circle Theorem). *Suppose $\mathscr{C}$ and $\mathscr{D}$ are two circles, and $\mathscr{D}$ contains a point in the interior of $\mathscr{C}$ and a point in the exterior of $\mathscr{C}$. Then there exist exactly two points where the circles intersect, one on each side of the line containing their centers.*

Theorem 12.11. *Suppose $\mathscr{C}(O_1, r_1)$ and $\mathscr{C}(O_2, r_2)$ are tangent to each other at P. Then O_1, O_2, and P are distinct and collinear, and $\mathscr{C}(O_1, r_1)$ and $\mathscr{C}(O_2, r_2)$ have a common tangent line at P.*

Theorem 12.12. *Every polygon inscribed in a circle is convex.*

Theorem 12.13 (Circumcircle Theorem). *A polygon $\mathscr{P}$ is cyclic if and only if the perpendicular bisectors of all its edges are concurrent. If this is the case, the point O where these perpendicular bisectors intersect is the unique circumcenter for $\mathscr{P}$, and the circle with center O and radius equal to the distance from O to any vertex is the unique circumcircle.*

Theorem 12.14. *Every triangle has a unique circumscribed circle.*

Theorem 12.15 (Incircle Theorem). *A convex polygon $\mathscr{P}$ has an inscribed circle if and only if the bisectors of all of its angles are concurrent. If this is the case, the point O where these bisectors intersect is the unique incenter for $\mathscr{P}$, and the circle with center O and radius equal to the distance from O to the line containing any edge is the unique incircle.*

Theorem 12.16. *Every triangle has a unique incircle.*

Exercise 12.7. *A parallelogram is cyclic if and only if it is a rectangle.*

Postulates Equivalent to the Euclidean Parallel Postulate

Euclidean Axiom 1 (Euclid's Fifth Postulate). *If ℓ and ℓ' are two lines cut by a transversal t in such a way that the measures of two consecutive interior angles add up to less than $180°$, then ℓ and ℓ' intersect on the side of t where those two angles lie.*

Euclidean Axiom 2 (The Euclidean Parallel Postulate). *For every line ℓ and for every point P that does not lie on ℓ, there is exactly one line m such that P lies on m and $m \parallel \ell$.*

Euclidean Axiom 3 (Hilbert's Parallel Postulate). *For every line ℓ and for every point P that does not lie on ℓ, there is at most one line m such that P lies on m and $m \parallel \ell$.*

Euclidean Axiom 4 (The Alternate Interior Angles Postulate). *If two parallel lines are cut by a transversal, then both pairs of alternate interior angles are congruent.*

Euclidean Axiom 5 (Proclus's Axiom). *If ℓ and ℓ' are parallel lines and t is a line that is distinct from ℓ but intersects ℓ, then t also intersects ℓ'.*

Euclidean Axiom 6 (Transitivity of Parallelism). *If ℓ, m, and n are distinct lines such that $\ell \parallel m$ and $m \parallel n$, then $\ell \parallel n$.*

Euclidean Axiom 7 (Wallis's Postulate). *Given any triangle $\triangle ABC$ and any positive real number r, there exists a triangle $\triangle DEF$ similar to $\triangle ABC$, with scale factor $r = DE/AB = EF/AC = EF/BC$.*

Euclidean Axiom 8 (The Angle-Sum Postulate). *The angle sum of every triangle is equal to $180°$.*

Euclidean Axiom 9 (Clairaut's Axiom). *There exists a rectangle.*

Euclidean Axiom 10 (Weak Angle-Sum Postulate). *There exists a triangle with zero defect.*

Euclidean Axiom 11 (Pythagorean Postulate). *If $\triangle ABC$ is a right triangle whose legs have lengths a and b, and whose hypotenuse has length c, then $a^2 + b^2 = c^2$.*

The Postulates of Hyperbolic Geometry

Axioms 1–9 of neutral geometry.

Axiom 10′ (The Hyperbolic Parallel Postulate). *For each line ℓ and each point P that does not lie on ℓ, there are at least two distinct lines m and n that contain P and are parallel to ℓ.*

Theorems of Hyperbolic Geometry

All of the theorems of neutral geometry.

Theorem 14.1 (Hyperbolic Angle-Sum Theorem). *In hyperbolic geometry, every convex polygon has positive defect.*

Corollary 14.2. *In hyperbolic geometry, there does not exist a rectangle.*

Theorem 14.3 (AAA Congruence Theorem). *In hyperbolic geometry, if there is a correspondence between the vertices of two triangles such that the corresponding angles are congruent, then the triangles are congruent.*

Theorem 14.4 (Properties of Saccheri Quadrilaterals). *Every Saccheri quadrilateral has the following properties:*

(a) *Its diagonals are congruent.*
(b) *Its summit angles are congruent and acute.*
(c) *Its midsegment is perpendicular to both the base and the summit.*
(d) *It is a parallelogram.*
(e) *It is a convex quadrilateral.*

Theorem 14.5 (Properties of Lambert Quadrilaterals). *Every Lambert quadrilateral has the following properties:*

(a) *Its fourth angle is acute.*
(b) *It is a parallelogram.*
(c) *It is a convex quadrilateral.*

Theorem 14.6. *In a Lambert quadrilateral, either side between two right angles is strictly shorter than its opposite side.*

Theorem 14.7. *In a Saccheri quadrilateral, the base is strictly shorter than the summit, and the midsegment is strictly shorter than either leg.*

Theorem 14.8. *Suppose ℓ and m are two different lines, and there are two distinct points on ℓ that lie on the same side of m and are equidistant from m. Then ℓ and m have a common perpendicular, and thus are parallel.*

Theorem 14.9. *Suppose ℓ and m are two different lines. No three distinct points on ℓ are equidistant from m.*

Theorem 14.10 (Uniqueness of Common Perpendiculars). *If ℓ and m are parallel lines that admit a common perpendicular, then the common perpendicular is unique.*

Theorem 14.11. *Suppose ℓ and m are parallel lines. If either of the following conditions is satisfied, then ℓ and m have a common perpendicular.*

(a) *There are two distinct points on ℓ that are equidistant from m.*
(b) *There is a transversal for ℓ and m that makes congruent alternate interior angles.*

Theorem 15.4. *If $\overrightarrow{AB} \mid \overrightarrow{CD}$, then $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$.*

Theorem 15.5 (Existence and Uniqueness of Asymptotic Rays). *Suppose $\overrightarrow{CD}$ is a ray, and A is a point not on $\overleftrightarrow{CD}$. Then there exists a unique ray starting at A and asymptotic to $\overrightarrow{CD}$.*

Theorem 15.10. *Asymptotically parallel lines do not admit common perpendiculars.*

Theorem 15.13. *Suppose ℓ is a line and P is a point not on ℓ. Let F be the foot of the perpendicular from P to ℓ. There are exactly two lines through P that are asymptotically parallel to ℓ, and they make equal acute angle measures with $\overleftrightarrow{PF}$. Every line through P that makes a larger angle measure with $\overleftrightarrow{PF}$ is ultraparallel to ℓ, and every line that makes a smaller angle measure is not parallel to ℓ.*

Theorem 15.14 (Ultraparallel Theorem). *Two lines are ultraparallel if and only if they admit a common perpendicular.*

Theorem 15.15. *If ℓ and m are ultraparallel lines, then the distance from ℓ to m attains its minimum at the point on the common perpendicular, and increases without bound in both directions.*

Theorem 15.16. *If ℓ and m are asymptotically parallel lines, then the distance from ℓ to m becomes arbitrarily small in the direction of their asymptotic rays, and increases without bound in the other direction.*