Math 445 **Geometry for Teachers** **Spring 2008**

Handout #2

This handout is meant to be read in place of Sections 6.6–6.10 in Venema. (We will come back and read those sections later.) You should read these pages after reading Venema's Section 6.5.

Consequences of the Euclidean Parallel Postulate

From this point on in our study of Euclidean geometry, we officially add the Euclidean Parallel Postulate to our list of axioms. Thus, in addition to the six axioms of neutral geometry, we assume the following:

Euclidean Parallel Postulate. *For every line ℓ and for every point P that does not lie on ℓ, there is exactly one line m such that P lies on m and $m \parallel \ell$.*

As you learned in Venema's Corollary 6.5.6, the axioms of neutral geometry are already sufficient to prove that given a line ℓ and a point $P \notin \ell$, there exists at least one line through P and parallel to ℓ. Thus the real content of the Euclidean Parallel Postulate is the statement that there is only one such line. We will see in this handout and in Venema's Chapter 7 that many familiar properties of Euclidean geometry follow from this postulate.

The first of these properties is a converse to the Alternate Interior Angles Theorem. It is Euclid's Proposition 29, the first one for which he makes use of his fifth postulate.

Theorem H2.1 (Converse to the Alternate Interior Angles Theorem). *If two parallel lines are cut by a transversal, then both pairs of alternate interior angles are congruent.*

Proof. Suppose ℓ and m are parallel lines cut by a transversal t, and let A and B denote the points where t intersects ℓ and m, respectively. Choose either pair of alternate interior angles and label them $\angle CAB$ and $\angle ABD$. (Fig. 1).

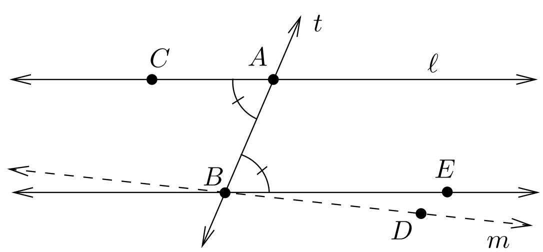

Figure 1: Proof of the converse to the Alternate Interior Angles Theorem.

By the Angle Construction Postulate, there is a ray $\overrightarrow{BE}$ on the same side of t as $\overrightarrow{BD}$ that makes an angle with $\overrightarrow{BA}$ that is congruent to $\angle CAB$. It follows from the Alternate Interior Angles Theorem that $\overleftrightarrow{BE}$ is parallel to ℓ. By the Euclidean Parallel Postulate, therefore, $\overleftrightarrow{BE}$ is equal to m. Since D and E are on the same side of t, this means that the rays $\overrightarrow{BE}$ and $\overrightarrow{BD}$ are equal, and therefore $\angle ABD = \angle ABE$. Since $\angle ABE$ is congruent to $\angle CAB$ by construction, we conclude that $\angle ABD \cong \angle CAB$. □

Corollary H2.2 (Converse to the Corresponding Angles Theorem). *If two parallel lines are cut by a transversal, then all four pairs of corresponding angles are congruent.*

Proof. Exercise H2.1. □

Corollary H2.3 (Converse to Corollary 6.5.5). *If two parallel lines are cut by a transversal, then each pair of interior angles lying on the same side of the transversal is supplementary.*

Proof. Exercise H2.2. ☐

These theorems lead to a number of additional properties of parallel lines, most of which will probably all seem familiar from your everyday experience. The proofs will be left as exercises.

The first result is a simple corollary of the Euclidean Parallel Postulate. Venema calls this ***Proclus's Axiom***, because a late Greek mathematician named Proclus used it as one step in a "proof" of Euclid's fifth postulate (which turned out, like so many others, to be incorrect).

Theorem H2.4 (Proclus's Lemma). *If ℓ and ℓ' are parallel lines and $t \neq \ell$ is a line such that t intersects ℓ, then t also intersects ℓ'.*

Proof. Exercise H2.3. ☐

The next theorem expresses the relationships between parallel and perpendicular lines in Euclidean geometry.

Theorem H2.5. *Suppose ℓ and ℓ' are parallel lines.*

(*a*) *If t is a transversal such that $t \perp \ell$, then $t \perp \ell'$.*

(*b*) *If m and n are lines such that $m \perp \ell$ and $n \perp \ell'$, then either $m = n$ or $m \parallel n$.*

Proof. Exercise H2.4. ☐

Finally, we have the following result, which seems so "obvious" that you might be tempted to think that it should follow immediately from the definition of parallel lines together with the axioms of neutral geometry. But, as we will see later in the course, it cannot be proved without the Euclidean Parallel Postulate (or something equivalent to it).

Theorem H2.6 (Transitivity of Parallelism). *If $\ell \parallel m$ and $m \parallel n$, then either $\ell = n$ or $\ell \parallel n$.*

Proof. Exercise H2.5. ☐

The Angle-Sum Theorem

The next theorem is one of the most important facts in Euclidean geometry. To state it concisely, we introduce the following terminology. If A, B, and C are noncollinear points, the ***angle sum*** for $\triangle ABC$ is the sum of the measures of its interior angles. It is denoted by $\sigma(\triangle ABC)$. More specifically, the angle sum is defined by the equation

$$\sigma(\triangle ABC) = \mu\angle CAB + \mu\angle ABC + \mu\angle BCA.$$

Theorem H2.7 (Angle-Sum Theorem). *If $\triangle ABC$ is a triangle, then $\sigma(\triangle ABC) = 180°$.*

Proof. By Corollary 6.5.6, there is a line m through A and parallel to $\overleftrightarrow{BC}$ (Fig. 2).

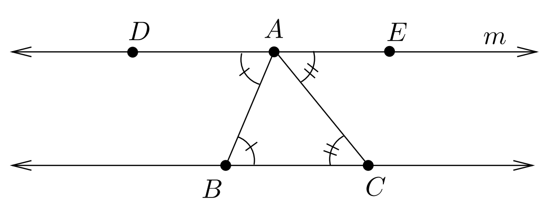

Figure 2: Proof of the Angle Sum Theorem.

We can choose points D, E on m such that D is on the opposite side of $\overleftrightarrow{AB}$ from C, and E is on the opposite side of $\overleftrightarrow{AC}$ from B. Then $\angle ABC$ and $\angle DAB$ form a pair of alternate interior angles for the transversal $\overleftrightarrow{AB}$, and $\angle BCA$ and $\angle CAE$ form a pair of alternate interior angles for the transversal $\overleftrightarrow{AC}$. It follows from the converse to the Alternate Interior Angles Theorem that $\angle ABC \cong \angle DAB$ and $\angle BCA \cong \angle CAE$.

It follows from the way we chose D and E that $\overrightarrow{AB}$ is between $\overrightarrow{AD}$ and $\overrightarrow{AC}$ (can you see how?). Therefore, by the Betweenness Theorem for Rays, we have $\mu\angle DAC = \mu\angle CAB + \mu\angle DAB$. On the other hand, because $\angle DAC$ and $\angle CAE$ form a linear pair, the Linear Pair Theorem implies that they are supplementary. Thus, combining all these results, we obtain

$$\begin{aligned}\mu\angle CAB + \mu\angle ABC + \mu\angle BCA &= \mu\angle CAB + \mu\angle DAB + \mu\angle CAE \\ &= \mu\angle DAC + \mu\angle CAE \\ &= 180^\circ.\end{aligned}$$

This completes the proof. □

Corollary H2.8. *In any triangle, the sum of the measures of any two angles is less than* 180°.

Proof. Since the measures of all three angles sum to 180°, and the measure of each angle is positive, the sum of any two of them must be strictly less than 180°. □

Corollary H2.9. *In any triangle, at least two of the angles are acute.*

Proof. Suppose at most one angle in a triangle is acute. Then the measures of two of the angles are at least 90°, so the sum of their measures is at least 180°, which contradicts the previous corollary. □

Corollary H2.10. *In any triangle, the measure of each exterior angle is equal to the sum of the measures of the two remote interior angles.*

Proof. This follows immediately from the Angle-Sum Theorem and the Linear Pair Theorem. □

One important application of the angle-sum theorem is to elucidate the relationship between Euclid's fifth postulate and the Euclidean Parallel Postulate. Note that the fifth postulate actually stated by Euclid did not refer explicitly to parallel lines; for that reason the postulate we call the Euclidean Parallel Postulate is sometimes referred to as ***Playfair's Postulate***, after an eighteenth-century Scottish mathematician named John Playfair, who proposed it as a more intuitive replacement for Euclid's fifth.

The next theorem shows that Euclid's fifth postulate follows from the Euclidean Parallel Postulate.

Theorem H2.11 (Euclid's Postulate V). *If ℓ and ℓ' are two lines cut by a transversal t in such a way that the sum of the measures of the two interior angles on one side of t is less than* 180°, *then ℓ and ℓ' intersect on that side of t.*

Proof. First note that ℓ and ℓ' are not parallel, because if they were, Corollary H2.3 would imply that two interior angles on the same side of t would have measures adding up to exactly 180°. Thus there is a point C where ℓ and ℓ' intersect. It remains only to show that C is on the same side of t as the two angles whose measures add up to less than 180°.

For definiteness, let us label the point where ℓ and t intersect as A, and the point where ℓ' and t intersect as B. Denote the two interior angles at A as $\angle 1$ and $\angle 2$, and those at B as $\angle 3$ and $\angle 4$, with the labels chosen so that $\angle 2$ and $\angle 4$ are on the same side of t as C, and $\angle 1$ and $\angle 3$ are on the other side (Fig. 3). Then Corollary H2.8 applied to $\triangle ABC$ implies that $\mu\angle 2 + \mu\angle 4 < 180^\circ$. On the other hand, because $\angle 1$ and $\angle 2$ form a linear pair, as do $\angle 3$ and $\angle 4$, a little algebra shows that

$$\begin{aligned}\mu\angle 1 + \mu\angle 3 &= (180^\circ - \mu\angle 2) + (180^\circ - \mu\angle 4) \\ &= 360^\circ - (\mu\angle 2 + \mu\angle 4) \\ &> 180^\circ.\end{aligned}$$

Thus the two interior angles whose measures add up to less than 180° can only be $\angle 2$ and $\angle 4$, and C is on the same side of t as these angles. □

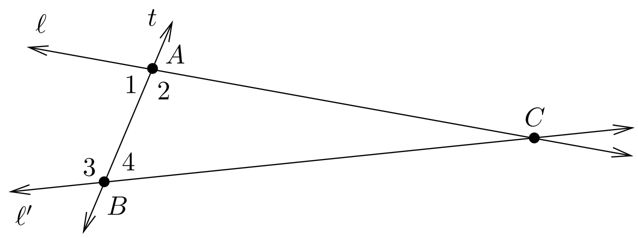

Figure 3: Euclid's fifth postulate.

In fact, the converse is also true: If in addition to the six postulates of Neutral Geometry, we assume Euclid's Postulate V instead of the Euclidean Parallel Postulate, then the Euclidean Parallel Postulate can be proved as a theorem. (See Exercise H2.6.) Thus, in the presence of the axioms of neutral geometry, the Euclidean Parallel Postulate and Euclid's Postulate V are ***equivalent***, meaning that each one implies the other. When we go back to Venema's Section 6.8, we will see that there are many more results that are also equivalent to the Euclidean Parallel Postulate.

Quadrilaterals

So far in our study of geometry, we have concentrated most of our attention on triangles. Almost as important as triangles are four-sided figures (quadrilaterals). In this section we describe some of the most important properties of such figures in the Euclidean setting.

Suppose A, B, C, and D are four distinct points with the following properties:

(a) No three of the points are collinear;

(b) If two of the segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ intersect, they do so only at a common endpoint.

Then the union of the four segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ is called a ***quadrilateral***, and is denoted by $\square ABCD$. Note that the order in which the points are listed is significant: If $\square ABCD$ is a quadrilateral, then some reorderings of the vertices, such as $\square BCDA$ and $\square DCBA$, represent the same quadrilateral (i.e., the union of the same four line segments), but other orderings, such as $\square ACBD$, do not. In fact, $\square ACBD$ might not represent a quadrilateral at all, because two of the segments might intersect at an interior point (Fig. 5).

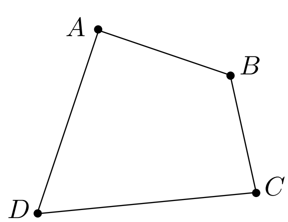

Figure 4: A quadrilateral.

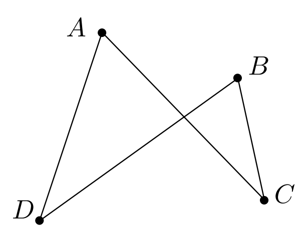

Figure 5: Not a quadrilateral.

Here is some standard terminology regarding quadrilaterals. Suppose $\square ABCD$ is a quadrilateral.

- The four points A, B, C, and D are called the ***vertices*** of $\square ABCD$.
- The four segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ are called the ***sides*** of $\square ABCD$.
- The two segments $\overline{AC}$ and $\overline{BD}$ are called the ***diagonals*** of $\square ABCD$.

- Any pair sides of $\square ABCD$ that do not intersect are called ***opposite sides***.
- Any pair of sides of $\square ABCD$ that intersect at a common endpoint are called ***adjacent sides***.
- The four angles formed by pairs of adjacent sides are called the ***angles of the quadrilateral***.
- Two quadrilaterals are said to be ***congruent*** if there is a correspondence between their vertices such that all four pairs of corresponding sides and all four pairs of corresponding angles are congruent. The notation $\square ABCD \cong \square A'B'C'D'$ means that $\square ABCD$ is congruent to $\square A'B'C'D'$ under the correspondence $A \leftrightarrow A'$, $B \leftrightarrow B'$, $C \leftrightarrow C'$, and $D \leftrightarrow D'$.
- $\square ABCD$ is a ***parallelogram*** if both pairs of opposite sides are parallel.
- $\square ABCD$ is a ***rectangle*** if all four of its angles are right angles.
- $\square ABCD$ is a ***rhombus*** if all four of its sides are congruent.
- $\square ABCD$ is a ***square*** if it is both a rhombus and a rectangle.

We wish to prove an analogue of the angle-sum theorem for quadrilaterals. It will say that the sum of the "interior angles" of a quadrilateral is equal to 360°. But there is a complication in defining interior angles for quadrilaterals that did not arise in the case of triangles. To see why, consider the quadrilateral pictured in Fig. 6. The two edges that meet at B form an angle, which is by definition one of the angles of $\square ABCD$.

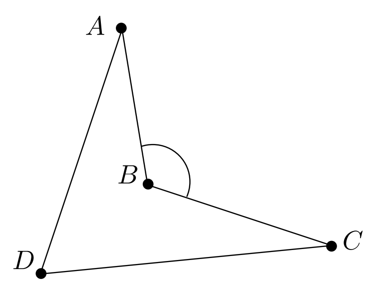

Figure 6: A nonconvex quadrilateral.

However, this angle is not the one we would want to consider as an "interior angle" of the quadrilateral.

It is possible to define what we mean by "interior" and "exterior" angles of a quadrilateral, and extend our notion of angle measures in such a way that in a quadrilateral like that in Fig. 6, the "interior angle" at B has a measure greater than 180°; with these conventions, the angle-sum theorem we are about to state would apply to such quadrilaterals as well. However, the definitions involve some intricate subtleties, and for the purposes we have in mind it is simpler just to rule out quadrilaterals of this type.

For that reason, following Venema, we make the following definition. A ***convex quadrilateral*** is one with the property that every vertex is contained in the interior of the angle formed by the two segments that do not contain that vertex. For example, the quadrilateral in Fig. 4 is convex; however, the one in Fig. 6 is not, because the vertex D is not in the interior of $\angle ABC$. Note that a convex quadrilateral is not the same thing as a *convex set*, as defined in Definition 5.5.1. (The definitions are related – if we had taken the trouble to define the "interior" of a quadrilateral, then we could show that $\square ABCD$ is a convex quadrilateral if and only if its interior is a convex set; but we will not do so.)

Here is the main result in this section. If $\square ABCD$ is a convex quadrilateral, we define its ***angle sum***, denoted by $\sigma(\square ABCD)$, to be the sum of the measures of its four angles:

$$\sigma(\square ABCD) = \mu\angle ABC + \mu\angle BCD + \mu\angle CDA + \mu\angle DAB.$$

Theorem H2.12 (Angle-Sum Theorem for Convex Quadrilaterals). *If $\square ABCD$ is a convex quadrilateral, then $\sigma(\square ABCD) = 360°$.*

Proof. Exercise H2.7. □

Because of the importance of the preceding theorem, it is useful to have some simple criteria for recognizing when a quadrilateral is convex. The next few theorems give several such criteria.

Theorem H2.13. *Every parallelogram is a convex quadrilateral.*

Proof. Exercise H2.8. □

Theorem H2.14 (Truncated Triangle Theorem). *Suppose $\triangle ABC$ is a triangle, and D and E are points such that $A * D * B$ and $A * E * C$. Then $\square BCED$ is a convex quadrilateral (Fig. 7).*

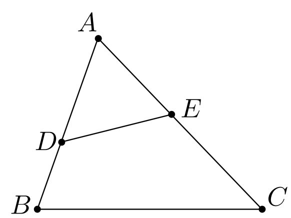

Figure 7: A truncated triangle is a convex quadrilateral.

Proof. Because E is between A and C, it follows from the Betweenness vs. Betweenness Theorem (Theorem 5.7.10) that $\overrightarrow{BE}$ is between $\overrightarrow{BA}$ and $\overrightarrow{BC}$, or in other words that E is in the interior of $\angle DBC$. The same argument shows that D is in the interior of $\angle ECB$.

To show that C is in the interior of $\angle BDE$, we note that the Y-Theorem (Corollary 5.7.7) applied to the ray $\overrightarrow{AC}$ shows that C is on the same side of $\overleftrightarrow{DB}$ as E. On the other hand, since A and B are on opposite sides of $\overleftrightarrow{DE}$, as are A and C, it follows that C is on the same side of $\overleftrightarrow{DE}$ as B. By definition, this means that C is in the interior of $\angle BDE$. Exactly the same argument shows that B is in the interior of $\angle CED$, and therefore $\square BCED$ is a convex quadrilateral. □

The third criterion is actually a necessary *and* sufficient condition for convexity.

Theorem H2.15. *The quadrilateral $\square ABCD$ is convex if and only if its diagonals have an interior point in common (Fig. 8).*

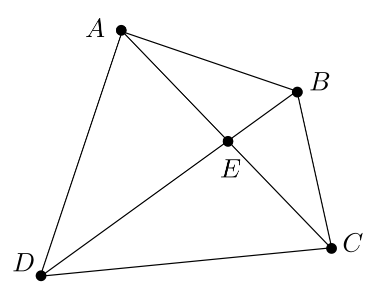

Figure 8: The diagonals of a convex quadrilateral have an interior point in common.

Proof. First assume that $\square ABCD$ is a quadrilateral whose diagonals $\overline{AC}$ and $\overline{BD}$ intersect in a point E that is interior to both diagonals. Because E is between A and C, it follows from the Betweenness vs. Betweenness Theorem (Theorem 5.7.10) that $\overrightarrow{BE}$ is between $\overrightarrow{BA}$ and $\overrightarrow{BC}$. Since D is a point on $\overrightarrow{BE}$, it is in the interior of $\angle ABC$ by Corollary 5.7.8. The same argument shows that each of the other three vertices is in the interior of the opposite angle.

Conversely, assume that $\square ABCD$ is a convex quadrilateral. Because D is in the interior of $\angle ABC$, it follows from the Crossbar Theorem (applied to $\triangle ABC$) that $\overrightarrow{BD}$ intersects the interior of $\overline{AC}$. Call the intersection point E. Similarly, because C is in the interior of $\angle BAD$, the Crossbar Theorem implies that $\overrightarrow{AC}$ intersects the interior of $\overline{BD}$ in a point E'. Because the lines $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ can have at most one point of intersection, it follows that $E = E'$, and therefore E is the required common interior point of the diagonals $\overline{AC}$ and $\overline{BD}$. □

Similar Triangles

Our last topic in this handout is a brief introduction to similar triangles. We say that two triangles are ***similar*** if there is a correspondence between their vertices such that corresponding angles are congruent. The notation $\triangle ABC \sim \triangle A'B'C'$ means that $\triangle ABC$ is similar to $\triangle A'B'C'$ under the correspondence $A \leftrightarrow A'$, $B \leftrightarrow B'$, and $C \leftrightarrow C'$, or more specifically that

$$\angle ABC \cong \angle A'B'C', \qquad \angle BCA \cong \angle B'C'A', \qquad \angle BCA \cong \angle B'C'A'.$$

The next theorem is a fundamental existence result in Euclidean geometry. Venema calls it ***Wallis's Postulate*** because the 17th-century English mathematician John Wallis proposed it as a replacement for Euclid's fifth postulate.

Theorem H2.16 (Wallis's Lemma). *If $\triangle ABC$ is a triangle and $\overline{DE}$ is a segment, then there exists a point F such that $\triangle ABC \sim \triangle DEF$.*

Proof. Given $\triangle ABC$ and any segment $\overline{DE}$, the Angle Construction Postulate ensures that there exists a point G such that $\angle EDG \cong \angle BAC$ (Fig. 9). Similarly, there exists a point H on the same side of $\overleftrightarrow{DE}$

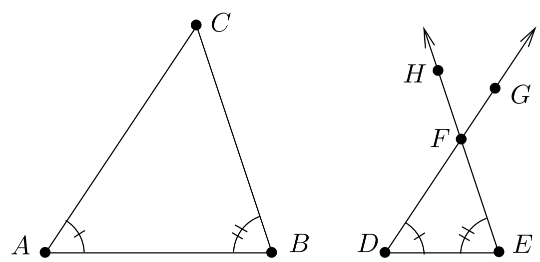

Figure 9: Proof of Wallis's Lemma.

as G such that $\angle DEH \cong \angle ABC$. Because the measures of $\angle ABC$ and $\angle BAC$ sum to less than 180° by Corollary H2.8, it follows that the measures of $\angle DEH$ and $\angle EDG$ also sum to less than 180°. By Euclid's Postulate V, therefore, there is a point F on the same side of $\overleftrightarrow{DE}$ as G and H where the lines $\overleftrightarrow{DG}$ and $\overleftrightarrow{EH}$ intersect. By the angle-sum theorem and subtraction,

$$\mu\angle DFE = 180^\circ - (\mu\angle DEF + \mu\angle EDF) = 180^\circ - (\mu\angle ABC + \mu\angle BAC) = \mu\angle BCA.$$

Thus $\triangle ABC \sim \triangle DEF$ by definition of similar triangles. □

Exercises

H2.1. Prove the converse to the Corresponding Angles Theorem (Corollary H2.2).

H2.2. Prove the converse to Corollary 6.5.5 (Corollary H2.3).

H2.3. Prove Proclus's Lemma (Theorem H2.4).

H2.4. Prove Theorem H2.5.

H2.5. Prove the transitivity of parallelism (Theorem H2.6).

H2.6. Prove that the six axioms of Neutral Geometry plus Euclid's Postulate V imply the Euclidean Parallel Postulate.

H2.7. Prove the Angle-Sum Theorem for Convex Quadrilaterals (Theorem H2.12). Where do you use the hypothesis that the quadrilateral is convex?

H2.8. Prove that every parallelogram is a convex quadrilateral (Theorem H2.13).

H2.9. Prove that every rectangle is a parallelogram.