Axioms of Neutral Geometry

The Existence Postulate. *The collection of all points forms a nonempty set. There is more than one point in that set.*

The Incidence Postulate. *Every line is a set of points. For every pair of distinct points A and B there is exactly one line ℓ such that $A \in \ell$ and $B \in \ell$.*

The Ruler Postulate. *For every pair of points P and Q there exists a real number PQ, called the* ***distance from P to Q****. For each line ℓ there is a one-to-one correspondence from ℓ to $\mathbb{R}$ such that if P and Q are points on the line that correspond to the real numbers x and y, respectively, then $PQ = |x - y|$.*

The Plane Separation Postulate. *For every line ℓ, the points that do not lie on ℓ form two disjoint, nonempty sets H_1 and H_2, called* ***half-planes bounded by ℓ*** *or* ***sides of ℓ****, such that the following conditions are satisfied.*

1. *Each of H_1 and H_2 is convex.*
2. *If $P \in H_1$ and $Q \in H_2$, then $\overline{PQ}$ intersects ℓ.*

The Protractor Postulate. *For every angle $\angle ABC$ there exists a real number $\mu\angle ABC$, called the* ***measure of $\angle ABC$****. For every half-rotation $\mathrm{HR}(A, O, B)$, there is a one-to-one correspondence g from $\mathrm{HR}(A, O, B)$ to the interval $[0, 180] \subset \mathbb{R}$, which sends $\overrightarrow{OA}$ to 0 and sends the ray opposite $\overrightarrow{OA}$ to 180, and such that if $\overrightarrow{OC}$ and $\overrightarrow{OD}$ are any two distinct, nonopposite rays in $\mathrm{HR}(A, O, B)$, then*

$$\mu\angle COD = \left| g(\overrightarrow{OD}) - g(\overrightarrow{OC}) \right|.$$

The Side-Angle-Side Postulate. *If $\triangle ABC$ and $\triangle DEF$ are two triangles such that $\overline{AB} \cong \overline{DE}$, $\angle ABC \cong \angle DEF$, and $\overline{BC} \cong \overline{EF}$, then $\triangle ABC \cong \triangle DEF$.*

The Neutral Area Postulate. *Associated with each polygonal region R there is a nonnegative number $\alpha(R)$, called the* ***area of R****, such that the following conditions are satisfied.*

1. (Congruence) *If two triangles are congruent, then their associated triangular regions have equal areas.*
2. (Additivity) *If R is the union of two nonoverlapping polygonal regions R_1 and R_2, then $\alpha(R) = \alpha(R_1) + \alpha(R_2)$.*

Theorems of Neutral Geometry

Theorem 5.3.7. *If ℓ and m are two distinct, nonparallel lines, then there exists exactly one point P such that P lies on both ℓ and m.*

Theorem 5.4.6. *If P and Q are any two points, then*

1. *$PQ = QP$,*
2. *$PQ \geq 0$, and*
3. *$PQ = 0$ if and only if $P = Q$.*

Corollary 5.4.7. *$A * C * B$ if and only if $B * C * A$.*

Theorem 5.4.14 (The Ruler Placement Theorem). *For every pair of distinct points P and Q, there is a coordinate function $f\colon \overleftrightarrow{PQ} \to \mathbb{R}$ such that $f(P) = 0$ and $f(Q) > 0$.*

Proposition 5.5.4. *Let ℓ be a line and let A and B be points that do not lie on ℓ. The points A and B are on the same side of ℓ if and only if $\overline{AB} \cap \ell = \varnothing$. The points A and B are on opposite sides of ℓ if and only if $\overline{AB} \cap \ell \neq \varnothing$.*

Theorem 5.5.10 (Pasch's Theorem). *Let $\triangle ABC$ be a triangle and let ℓ be a line such that none of A, B, and C lies on ℓ. If ℓ intersects $\overline{AB}$, then ℓ also intersects either $\overline{AC}$ or $\overline{BC}$.*

Theorem A.1 (Betweenness Theorem for Points). *Suppose A, B, and C are distinct points all lying on a single line ℓ. Then the following statements are equivalent:*

(*a*) $AB + BC = AC$ (*i.e.,* $A * B * C$).

(*b*) *B lies in the interior of the line segment* $\overline{AC}$.

(*c*) *B lies on the ray* $\overrightarrow{AC}$ *and* $AB < AC$.

(*d*) *For any coordinate function* $f : \ell \to \mathbb{R}$, *the coordinate* $f(B)$ *is between* $f(A)$ *and* $f(C)$.

Corollary A.2. *If A, B, and C are three distinct collinear points, then exactly one of them lies between the other two.*

Theorem A.3 (Existence and Uniqueness of Midpoints) *Every line segment has a unique midpoint.*

Theorem A.4 (Ray Theorem) *Suppose A and B are distinct points, and f is a coordinate function for the line* $\overleftrightarrow{AB}$ *satisfying* $f(A) = 0$. *Then a point* $P \in \overleftrightarrow{AB}$ *is an interior point of* $\overrightarrow{AB}$ *if and only if its coordinate has the same sign as that of B.*

Corollary A.5. *If A and B are distinct points, and f is a coordinate function for the line* $\overleftrightarrow{AB}$ *satisfying* $f(A) = 0$ *and* $f(B) > 0$, *then* $\overrightarrow{AB} = \{P \in \overleftrightarrow{AB} : f(P) \geq 0\}$.

Corollary A.6. *If A, B, and C are distinct collinear points, then* $\overrightarrow{AB}$ *and* $\overrightarrow{AC}$ *are opposite rays if and only if* $B * A * C$, *and otherwise they are equal.*

Corollary A.7 (Segment Construction Theorem) *If* $\overline{AB}$ *is a line segment and* $\overrightarrow{CD}$ *is a ray, there is a unique interior point* $E \in \overrightarrow{CD}$ *such that* $\overline{CE} \cong \overline{AB}$.

Theorem A.8 (The Y-Theorem) *Suppose* ℓ *is a line, A is a point on* ℓ, *and B is a point not on* ℓ. *Then every interior point of* $\overrightarrow{AB}$ *is on the same side of* ℓ *as B.*

Theorem A.10. *If* $\angle ABC$ *is any angle, then* $0^\circ < \mu\angle ABC < 180^\circ$.

Theorem A.11 (Angle Construction Theorem) *Let A, O, and B be noncollinear points. For every real number m such that* $0 < m < 180$, *there is a unique ray* $\overrightarrow{OC}$ *with vertex O and lying on the same side of* $\overleftrightarrow{OA}$ *as B such that* $\mu\angle AOC = m^\circ$.

Theorem A.12 (Linear Pair Theorem) *If two angles form a linear pair, they are supplementary.*

Theorem A.13 (Vertical Angles Theorem) *Vertical angles are congruent.*

Theorem A.14 (Four Right Angles Theorem) *If* $\ell \perp m$, *then* ℓ *and m form four right angles.*

Theorem A.15 (Existence and Uniqueness of Perpendicular Bisectors) *Every line segment has a unique perpendicular bisector.*

Theorem A.16 (Betweenness vs. Betweenness) *Let A, O, and C be three noncollinear points and let B be a point on the line* $\overleftrightarrow{AC}$. *The point B is between points A and C if and only if the ray* $\overrightarrow{OB}$ *is between rays* $\overrightarrow{OA}$ *and* $\overrightarrow{OC}$.

Theorem A.18 (Betweenness Theorem for Rays) *Suppose O, A, B, and C are four distinct points such that no two of the rays* $\overrightarrow{OA}$, $\overrightarrow{OB}$, *and* $\overrightarrow{OC}$ *are equal and no two are opposite. Then the following statements are equivalent:*

(*a*) $\mu\angle AOB + \mu\angle BOC = \mu\angle AOC$.

(*b*) $\overrightarrow{OB}$ *lies in the interior of* $\angle AOC$ (*i.e.,* $\overrightarrow{OA} * \overrightarrow{OB} * \overrightarrow{OC}$).

(*c*) $\overrightarrow{OB}$ *lies in the half-rotation* $\mathrm{HR}(A, O, C)$ *and* $\mu\angle AOB < \mu\angle AOC$.

(*d*) $\overrightarrow{OA}$, $\overrightarrow{OB}$, *and* $\overrightarrow{OC}$ *all lie in some half-rotation, and if g is the coordinate function cooresponding to any such half-rotation, the coordinate* $g(\overrightarrow{OB})$ *is between* $g(\overrightarrow{OA})$ *and* $g(\overrightarrow{OC})$.

Corollary A.19. *If* $\overrightarrow{OA}$, $\overrightarrow{OB}$, *and* $\overrightarrow{OC}$ *are three rays that all lie on one half-rotation and such that no two are equal and no two are opposite, then exactly one is between the other two.*

Corollary from class. *If* $\overrightarrow{OA} * \overrightarrow{OB} * \overrightarrow{OC}$, *then A and B are on opposite sides of* $\overleftrightarrow{OC}$.

Theorem A.20 (Existence and Uniqueness of Angle Bisectors) *Every angle has a unique angle bisector.*

Theorem A.21 (The Crossbar Theorem) *If $\triangle ABC$ is a triangle and $\overrightarrow{AD}$ is a ray between $\overrightarrow{AB}$ and $\overrightarrow{AC}$, then $\overrightarrow{AD}$ intersects $\overline{BC}$.*

Theorem 5.8.5 (Isosceles Triangle Theorem). *If $\triangle ABC$ is a triangle and $\overline{AB} \cong \overline{AC}$, then $\angle ABC \cong \angle ACB$.*

Theorem 6.2.1 (ASA). *If $\triangle ABC$ and $\triangle DEF$ are two triangles such that $\angle CAB \cong \angle FDE$, $\overline{AB} \cong \overline{DE}$, and $\angle ABC \cong \angle DEF$, then $\triangle ABC \cong \triangle DEF$.*

Theorem 6.2.2 (Converse to the Isosceles Triangle Theorem). *If $\triangle ABC$ is a triangle such that $\angle ABC \cong \angle ACB$, then $\overline{AB} \cong \overline{AC}$.*

Exercise 6.3 (Construction of Perpendiculars). *For every line ℓ and for every point P that lies on ℓ, there exists a unique line m such that P lies on m and $m \perp \ell$.*

Theorem 6.2.3 (Existence of Perpendicular from an External Point). *For every line ℓ and for every external point P, there exists a line m such that P lies on m and $m \perp \ell$.*

Theorem 6.2.4 (Copying a Triangle). *If $\triangle ABC$ is a triangle, $\overline{DE}$ is a segment such that $\overline{DE} \cong \overline{AB}$, and H is a half-plane bounded by $\overleftrightarrow{DE}$, then there is a unique point $F \in H$ such that $\triangle DEF \cong \triangle ABC$.*

Theorem 6.3.2 (Exterior Angle Theorem). *The measure of an exterior angle for a triangle is strictly greater than the measure of either remote interior angle.*

Corollary 6.3.3 (Uniqueness of Perpendiculars). *For every line ℓ and for every external point P, there exists exactly one line m such that P lies on m and $m \perp \ell$.*

Theorem 6.3.4 (AAS). *If $\triangle ABC$ and $\triangle DEF$ are two triangles such that $\angle ABC \cong \angle DEF$, $\angle BCA \cong \angle EFD$, and $\overline{AC} \cong \overline{DF}$, then $\triangle ABC \cong \triangle DEF$.*

Theorem 6.3.6 (Hypotenuse-Leg Theorem). *If $\triangle ABC$ and $\triangle DEF$ are two right triangles with right angles at the vertices C and F, respectively, $\overline{AB} \cong \overline{DE}$, and $\overline{BC} \cong \overline{EF}$, then $\triangle ABC \cong \triangle DEF$.*

Theorem 6.3.7 (SSS). *If $\triangle ABC$ and $\triangle DEF$ are two triangles such that $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, $\overline{CA} \cong \overline{FD}$, then $\triangle ABC \cong \triangle DEF$.*

Theorem 6.4.1 (Scalene Inequality). *Let A, B, and C be three noncollinear points. Then $AB > BC$ if and only if $\mu(\angle ACB) > \mu(\angle BAC)$.*

Theorem 6.4.2 (Triangle Inequality). *If A, B, and C are three noncollinear points, then $AC < AB + BC$.*

Theorem 6.4.3 (Hinge Theorem). *If $\triangle ABC$ and $\triangle DEF$ are two triangles such that $AB = DE$, $AC = DF$, and $\mu(\angle BAC) < \mu(\angle EDF)$, then $BC < EF$.*

Theorem 6.4.4. *Let ℓ be a line, let P be an external point, and let F be the foot of the perpendicular from P to ℓ. If R is any point on line ℓ that is different from F, then $PR > PF$.*

Lemma from class (Interior Foot Lemma). *In $\triangle ABC$, if $\angle A$ and $\angle B$ are acute, then the foot of the perpendicular from C to $\overleftrightarrow{AB}$ lies in the interior of $\overleftrightarrow{AB}$.*

Theorem 6.4.6 (Pointwise Characterization of Angle Bisector). *Let A, B, and C be three noncollinear points and let P be a point in the interior of $\angle BAC$. Then P lies on the angle bisector of $\angle BAC$ if and only if $d(P, \overleftrightarrow{AB}) = d(P, \overleftrightarrow{AC})$.*

Theorem 6.4.7 (Pointwise Characterization of Perpendicular Bisector). *Let A and B be distinct points. A point P lies on the perpendicular bisector of $\overline{AB}$ if and only if $PA = PB$.*

Theorem 6.5.2 (Alternate Interior Angles Theorem). *If ℓ and ℓ' are two lines cut by a transversal t in such a way that a pair of alternate interior angles is congruent, then ℓ is parallel to ℓ'.*

Corollary 6.5.4 (Corresponding Angles Theorem). *If ℓ and ℓ' are two lines cut by a transversal t in such a way that two corresponding angles are congruent, then ℓ is parallel to ℓ'.*

Corollary 6.5.5 (Supplementary Angles Theorem). *If ℓ and ℓ' are two lines cut by a transversal t in such a way that two nonalternating angles on the same side of t are supplements, then ℓ is parallel to ℓ'.*

Corollary 6.5.6 (Existence of Parallels). *If ℓ is a line and P is an external point, then there is a line m such that P lies on m and m is parallel to ℓ.*

Addendum (Existence of a Parallel with a Common Perpendicular). *If ℓ is a line and P is an external point, then there is a line m that is parallel to ℓ and contains P, and a line t through P that is a common perpendicular for ℓ and m.*

Corollary 6.5.8. (Common Perpendicular Theorem). *If ℓ and ℓ' are distinct lines that admit a common perpendicular, then they are parallel.*

Theorem 6.6.2 (Saccheri–Legendre Theorem). *If $\triangle ABC$ is any triangle, then $\sigma(\triangle ABC) \leq 180°$.*

Theorem 6.9.2 (Additivity of Defect).

1. *If $\triangle ABC$ is a triangle and E is a point in the interior of $\overline{BC}$, then $\delta(\triangle ABC) = \delta(\triangle ABE) + \delta(\triangle ECA)$.*
2. *If $\square ABCD$ is a convex quadrilateral, then $\delta(\square ABCD) = \delta(\triangle ABC) + \delta(\triangle ACD)$.*

Theorem 6.9.10 (Properties of Saccheri quadrilaterals). *If $\square ABCD$ is a Saccheri quadrilateral with base $\overline{AB}$, then*

1. *the diagonals $\overline{AC}$ and $\overline{BD}$ are congruent,*
2. *the summit angles $\angle BCD$ and $\angle ADC$ are congruent,*
3. *the segment joining the midpoint of $\overline{AB}$ to the midpoint of $\overline{CD}$ is perpendicular to both $\overline{AB}$ and $\overline{CD}$,*
4. *$\square ABCD$ is a parallelogram,*
5. *$\square ABCD$ is a convex quadrilateral,*
6. *the summit angles $\angle BCD$ and $\angle ADC$ are acute.*

Theorem 6.9.11 (Properties of Lambert quadrilaterals). *If $\square ABCD$ is a Lambert quadrilateral with right angles at vertices A, B, and C, then*

1. *$\square ABCD$ is a parallelogram,*
2. *$\square ABCD$ is a convex quadrilateral, and*
3. *$\angle ADC$ is acute.*

Theorem 6.10.1 (The Universal Hyperbolic Theorem). *In every model of neutral geometry, either the Euclidean parallel postulate or the hyperbolic parallel postulate holds.*

Axioms of Euclidean Geometry

The Seven Postulates of Neutral Geometry.

The Euclidean Parallel Postulate. *For every line ℓ and for every point P that does not lie on ℓ, there is exactly one line m such that P lies on m and $m \parallel \ell$.*

The Euclidean Area Postulate. *If R is a rectangular region, then $\alpha(R) = \text{length}(R) \times \text{width}(R)$.*

Theorems of Euclidean Geometry

(All the theorems of neutral geometry are valid in Euclidean geometry.)

Theorem B.2 (Converse to the Alternate Interior Angles Theorem). *If two parallel lines are cut by a transversal, then both pairs of alternate interior angles are congruent.*

Corollary B.3 (Converse to the Corresponding Angles Theorem). *If two parallel lines are cut by a transversal, then all four pairs of corresponding angles are congruent.*

Corollary B.4 (Converse to the Supplementary Angles Theorem). *If two parallel lines are cut by a transversal, then each pair of interior angles lying on the same side of the transversal is supplementary.*

Theorem B.5 (Proclus's Lemma). *If ℓ and ℓ' are parallel lines and $t \neq \ell$ is a line such that t intersects ℓ, then t also intersects ℓ'.*

Theorem B.6 (Parallels and Perpendiculars) *Suppose ℓ and ℓ' are parallel lines.*

(*a*) *If t is a transversal such that $t \perp \ell$, then $t \perp \ell'$.*
(*b*) *If m and m' are distinct lines such that $m \perp \ell$ and $m' \perp \ell'$, then $m \parallel m'$.*

Theorem B.7 (Transitivity of Parallelism). *If ℓ, m, and n are distinct lines such that $\ell \parallel m$ and $m \parallel n$, then $\ell \parallel n$.*

Theorem B.8 (Angle-Sum Theorem). *If $\triangle ABC$ is a triangle, then $\sigma(\triangle ABC) = 180°$.*

Corollary B.9. *In any triangle, the sum of the measures of any two interior angles is less than $180°$.*

Corollary B.10. *In any triangle, at least two of the angles are acute.*

Corollary B.11. *In any triangle, the measure of each exterior angle is equal to the sum of the measures of the two remote interior angles.*

Theorem B.12 (The Euclidean Parallel Postulate Implies Euclid's Postulate V) *If ℓ and ℓ' are two lines cut by a transversal t in such a way that the sum of the measures of the two interior angles on one side of t is less than $180°$, then ℓ and ℓ' intersect on that side of t.*

Theorem B.13 (Euclid's Postulate V Implies the Euclidean Parallel Postulate). *The six axioms of Neutral Geometry together with Euclid's Postulate V imply the Euclidean Parallel Postulate.*

Theorem B.14 (Angle-Sum Theorem for Convex Quadrilaterals). *If $\square ABCD$ is a convex quadrilateral, then $\sigma(\square ABCD) = 360°$.*

Theorem B.15 (Truncated Triangle Theorem). *Suppose $\triangle ABC$ is a triangle, and D and E are points such that $A * D * B$ and $A * E * C$. Then $\square BCED$ is a convex quadrilateral.*

Theorem from class. *A quadrilateral is convex if and only if both pairs of opposite sides are semiparallel.*

Theorem B.16. *Every trapezoid is a convex quadrilateral.*

Corollary B.17. *Every parallelogram is a convex quadrilateral.*

Theorem B.18. *A quadrilateral is convex if and only if its diagonals intersect. If they do intersect, then the intersection point is an interior point of both diagonals.*

Theorem B.19. *Every parallelogram has the following properties.*

(*a*) *Both pairs of opposite sides are congruent.*
(*b*) *Both pairs of opposite angles are congruent.*
(*c*) *Its diagonals bisect each other.*

Theorem B.20. *Every rectangle has the following properties.*

(*a*) *It is a parallelogram.*
(*b*) *Its diagonals are congruent.*

Theorem B.21. *Every rhombus has the following properties.*

(*a*) *It is a parallelogram.*
(*b*) *Its diagonals intersect perpendicularly.*

Theorem 9.1.7. *If $\triangle ABC$ is a triangle and E is a point on the interior of $\overline{AC}$, then $\blacktriangle ABC = \blacktriangle ABE \cup \blacktriangle EBC$. Furthermore, $\blacktriangle ABE$ and $\blacktriangle EBC$ are nonoverlapping regions. Thus $\alpha(\triangle ABC) = \alpha(\triangle ABE) + \alpha(\triangle EBC)$.*

Exercise 9.3. *Let $\square ABCD$ be a convex quadrilateral. Then $\blacktriangle ABC \cup \blacktriangle CDA = \blacktriangle DAB \cup \blacktriangle BCD$, and each pair of triangles is nonoverlapping. Thus $\alpha(\square ABCD) = \alpha(\triangle ABC) + \alpha(\triangle CDA) = \alpha(\triangle DAB) + \alpha(\triangle BCD)$.*

Theorem 9.2.5. *The area of a triangular region is one-half the length of the base times the height.*

Exercise 9.8. *The area of a parallelogram is the length of the base times the height.*

Exercise 9.11. *The area of a trapezoid is the height times the average of the lengths of the bases.*

Theorem 9.2.8 (The Pythagorean Theorem). *Suppose $\triangle ABC$ is a right triangle with right angle $\angle C$, and let a, b, and c denote the lengths of the sides opposite A, B, and C, respectively. Then $a^2 + b^2 = c^2$.*

Theorem C.12 (Converse to the Pythagorean Theorem). *Suppose $\triangle ABC$ is a triangle, and let a, b, and c denote the lengths of the sides opposite A, B, and C, respectively. If $a^2 + b^2 = c^2$, then $\angle C$ is a right angle.*

Theorem C.1 (AA Similarity Theorem). *If $\triangle ABC$ and $\triangle DEF$ are triangles such that $\angle A \cong \angle D$ and $\angle B \cong \angle E$, then $\triangle ABC \sim \triangle DEF$.*

Theorem C.2 (Similar Triangle Construction Theorem). *If $\triangle ABC$ is a triangle, $\overline{DE}$ is a segment, and H is a half-plane bounded by $\overleftrightarrow{DE}$, then there is a unique point $F \in H$ such that $\triangle ABC \sim \triangle DEF$.*

Lemma C.3 (Sliding Lemma). *Suppose $\triangle ABC$ and $\triangle A'BC$ are two distinct triangles that have a common side $\overline{BC}$, such that $\overleftrightarrow{AA'} \parallel \overleftrightarrow{BC}$. Then $\alpha(\triangle ABC) = \alpha(\triangle A'BC)$.*

Lemma C.4. *Suppose $\triangle ABC$ is a triangle, and D is a point such that $B * D * C$. Then*

$$\frac{\alpha(\triangle ABD)}{\alpha(\triangle ABC)} = \frac{BD}{BC}.$$

Theorem C.5 (The Side-Splitter Theorem). *Suppose $\triangle ABC$ is a triangle, and ℓ is a line parallel to $\overleftrightarrow{BC}$ that intersects $\overline{AB}$ at an interior point D. Then ℓ also intersects $\overline{AC}$ at an interior point E, and*

$$\frac{AD}{AB} = \frac{AE}{AC}.$$

Theorem C.6 (Fundamental Theorem on Similar Triangles). *If $\triangle ABC \sim \triangle DEF$, then*

$$\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}. \tag{0.1}$$

Corollary C.7. *If $\triangle ABC \sim \triangle DEF$, then there is a positive number r such that*

$$AB = r \cdot DE, \quad AC = r \cdot DF, \quad BC = r \cdot EF.$$

Theorem C.8 (SAS Similarity Theorem). *If $\triangle ABC$ and $\triangle DEF$ are triangles such that $\angle A \cong \angle D$ and $AB/DE = AC/DF$, then $\triangle ABC \sim \triangle DEF$.*

Theorem C.9 (SSS Similarity Theorem). *If $\triangle ABC$ and $\triangle DEF$ are triangles such that $AB/DE = AC/DF = BC/EF$, then $\triangle ABC \sim \triangle DEF$.*

Theorem C.10 (Area Scaling Theorem). *If two triangles are similar, then the ratio of their areas is the square of the ratio of any two corresponding sides; that is, if $\triangle ABC \sim \triangle DEF$ and $AB = r \cdot DE$, then $\alpha(\triangle ABC) = r^2 \cdot \alpha(\triangle DEF)$.*

Theorem 10.2.1. *If γ is a circle and ℓ is a line, then the number of points in $\gamma \cap \ell$ is 0, 1, or 2.*

Theorem 10.2.4 (Tangent Line Theorem). *Let t be a line, $\gamma = \mathcal{C}(O, r)$ a circle, and P a point of $t \cap \gamma$. The line t is tangent to the circle γ at the point P if and only if $\overleftrightarrow{OP} \perp t$.*

Theorem 10.2.5. *If γ is a circle and t is a tangent line that meets γ at P, then every point of t except for P is outside γ.*

Theorem 10.2.6 (Secant Line Theorem). *If $\gamma = \mathcal{C}(O, r)$ is a circle and ℓ is a secant line that intersects γ at distinct points P and Q, then O lies on the perpendicular bisector of the chord $\overline{PQ}$.*

Theorem 10.2.7. *If γ is a circle and ℓ is a secant line such that ℓ intersects γ at points P and Q, then every point on the interior of $\overline{PQ}$ is inside γ and every point of $\ell \setminus \overline{PQ}$ is outside γ.*

Theorem from class. *If γ and γ' are two distinct circles, then the number of points in $\gamma \cap \gamma'$ is 0, 1, or 2.*

Theorem 10.2.12 (Tangent Circles Theorem). *If the circles $\gamma_1 = \mathcal{C}(O_1, r_1)$ and $\gamma_2 = \mathcal{C}(O_2, r_2)$ are tangent at P, then the centers O_1 and O_2 are distinct and the three points O_1, O_2, and P are collinear. Furthermore, the circles share a common tangent line at P.*

Theorem from class (Circle-Line Theorem). *If γ is a circle and ℓ is a line that contains a point inside γ, then ℓ is a secant line for γ.*

Theorem from class (Converse to the Triangle Inequality). *If a, b, and c are three positive real numbers such that each one is less than the sum of the other two, then there exists a triangle whose side lengths are a, b, and c.*

Theorem from class (Two Circles Theorem). *Let γ and γ' be two distinct circles. If there exists a point that lies on γ' and is inside γ, and there exists another point that lies on γ' and is outside γ, then $\gamma \cap \gamma'$ consists of exactly two points.*

Theorem 10.3.2 (Circumscribed Circle Theorem). *Every Euclidean triangle has a unique circumscribed circle. The three perpendicular bisectors of the sides of any triangle are concurrent and meet at the circumcenter of the triangle.*

Theorem 10.3.8 (Inscribed Circle Theorem). *Every triangle has a unique inscribed circle. The bisectors of the interior angles in any triangle are concurrent and the point of concurrency is the incenter of the triangle.*

Theorem 10.4.1. *Let $\triangle ABC$ be a triangle and let M be the midpoint of $\overline{AB}$. If $AM = MC$, then $\angle ACB$ is a right angle.*

Corollary 10.4.2 (An angle inscribed in a semicircle is a right angle). *If the vertices of triangle $\triangle ABC$ lie on a circle and $\overline{AB}$ is a diameter of that circle, then $\angle ACB$ is a right angle.*

Theorem 10.4.3. *Let $\triangle ABC$ be a triangle and let M be the midpoint of $\overline{AB}$. If $\angle ACB$ is a right angle, then $AM = MC$.*

Corollary 10.4.4 (Converse to Corollary 10.4.2). *If $\angle ACB$ is a right angle, then $\overline{AB}$ is a diameter of the circle that circumscribes $\triangle ABC$.*

Theorem 10.4.5 (The 30-60-90 Theorem). *If the interior angles in triangle $\triangle ABC$ measure $30°$, $60°$, and $90°$, then the length of the side opposite the $30°$ angle is one half the length of the hypotenuse.*

Theorem 10.4.6 (Converse to the 30-60-90 Theorem). *If $\triangle ABC$ is a right triangle such that the length of one leg is one-half the length of the hypotenuse, then the interior angles of the triangle measure $30°$, $60°$, and $90°$.*

Theorem 10.6.6. *If $\mathcal{C}(O, R)$ and $\mathcal{C}(O', r')$ are two circles, and C, C' are their respective circumferences, then $C/r = C'/r'$. Thus there is a universal constant π such that every circle of radius r has circumference $2\pi r$.*

Theorem from class. *If $\mathcal{C}(O, R)$ and $\mathcal{C}(O', r')$ are two circles, and A and A' are their respective areas, then $A/r^2 = A'/{r'}^2$. Thus there is a universal constant k such that every circle of radius r has area kr^2.*

Theorem 10.6.11 (Archimedes' Theorem). *If γ is a circle of radius r, C is the circumference of γ, and A is the area of the associated circular region, then $A = \frac{1}{2}rC$.*

Corollary 10.6.12. *The area of every circle of radius r is πr^2.*

Theorem 12.2.6. *The composition of two isometries is an isometry. The inverse of an isometry is an isometry.*

Theorem 12.2.7 (Properties of Isometries). *Let $T\colon \mathbb{P} \to \mathbb{P}$ be an isometry. Then T preserves the following geometric relationships.*

1. *T preserves collinearity; that is, if P, Q, and R are three collinear points, then $T(P)$, $T(Q)$, and $T(R)$ are collinear.*
2. *T preserves betweenness of points; that is, if P, Q, and R are three collinear points such that $P * Q * R$, then $T(P) * T(Q) * T(R)$.*
3. *T preserves segments and their lengths; that is, if A and B are points and A' and B' are their images under T, then $T(\overline{AB}) = \overline{A'B'}$ and $\overline{A'B'} \cong \overline{AB}$.*
4. *T preserves lines; that is, if ℓ is a line, then $T(\ell)$ is a line.*
5. *T preserves betweenness of rays; that is, if $\overrightarrow{OP}$, $\overrightarrow{OQ}$, and $\overrightarrow{OR}$ are three rays such that $\overrightarrow{OP}$ is between $\overrightarrow{OQ}$ and $\overrightarrow{OR}$, then $\overrightarrow{O'P'}$ is between $\overrightarrow{O'Q'}$ and $\overrightarrow{O'R'}$.*
6. *T preserves angles and their measures; that is, if $\angle BAC$ is an angle, then $T(\angle BAC)$ is an angle and $T(\angle BAC) \cong \angle BAC$.*
7. *T preserves triangles and their measures; that is, if $\triangle BAC$ is a triangle, then $T(\triangle BAC)$ is a triangle and $T(\triangle BAC) \cong \triangle BAC$.*
8. *T preserves circles and their radii; that is, if γ is a circle with center O and radius r, then $T(\gamma)$ is a circle with center $T(O)$ and radius r.*
9. *T preserves polygonal regions and their areas; that is, if R is a polygonal region, then $T(R)$ is a polygonal region and $\alpha(T(R)) = \alpha(R)$.*
10. **(added in class).** *T preserves half-planes; that is, if ℓ is a line and P and Q are points not on ℓ, then $T(P)$ and $T(Q)$ are on the same side of $T(\ell)$ if and only if P and Q are on the same side of ℓ.*

Theorem 12.2.8 (Fundamental Theorem of Isometries). *If $\triangle ABC$ and $\triangle DEF$ are two triangles with $\triangle ABC \cong \triangle DEF$, then there exists a unique isometry T such that $T(A) = D$, $T(B) = E$, and $T(C) = F$.*

Corollary 12.2.9 (An Isometry is Determined by Its Action on Three Noncollinear Points). *If f and g are two isometries and A, B, and C are three noncollinear points such that $f(A) = g(A)$, $f(B) = g(B)$, and $f(C) = g(C)$, then $f(P) = g(P)$ for every point P.*

Corollary 12.2.11. *Every isometry of the plane can be expressed as a composition of reflections. The number of reflections required is at most three.*

Theorem 12.3.4 (First Rotation Theorem). *An isometry is a rotation if and only if it is a composition of reflections through two nonparallel lines.*

Theorem 12.3.5 (First Translation Theorem). *An isometry is a translation if and only if it is a composition of reflections through two lines that are either identical or parallel.*

Theorem 12.4.7 (Classification of Euclidean Motions). *Every Euclidean motion is either the identity, a reflection, a rotation, a translation, or a glide reflection.*

Axioms of Hyperbolic Geometry

The Seven Postulates of Neutral Geometry.

The Hyperbolic Parallel Postulate. *For every line ℓ and for every point P that does not lie on ℓ, there are at least two lines m and n such that P lies on both m and n and both m and n are parallel to ℓ.*

Theorems of Hyperbolic Geometry

(All the theorems of neutral geometry are valid in hyperbolic geometry.)

Theorem 8.2.1 (Triangle Angle-Sums in Hyperbolic Geometry). *For every triangle $\triangle ABC$, $\sigma(\triangle ABC) < 180^\circ$.*

Theorem 8.2.1 (Quadrilateral Angle-Sums in Hyperbolic Geometry). *For every quadrilateral* $\square ABCD$, $\sigma(\square ABCD) < 360°$.

Theorem 8.2.3. *There does not exist a rectangle.*

Corollary 8.2.4 (Positivity of Defect). *For every triangle* $\triangle ABC$, $0° < \delta(\triangle ABC) < 180°$.

Theorem 8.2.7. *In a Lambert quadrilateral, the length of a side between two right angles is strictly less than the length of the opposite side.*

Corollary 8.2.9. *In a Saccheri quadrilateral, the length of the altitude is less than the length of a side.*

Corollary 8.2.10 *In a Saccheri quadrilateral, the length of the summit is greater than the length of the base.*

Theorem 8.2.11 (AAA Congruence Theorem). *If* $\triangle ABC$ *is similar to* $\triangle DEF$, *then* $\triangle ABC$ *is congruent to* $\triangle DEF$.

Theorem 8.3.1. *If* ℓ *is a line,* P *is an external point, and* m *is a line such that* P *lies on* m, *then there exists at most one point* Q *such that* $Q \neq P$, Q *lies on* m, *and* $d(Q, \ell) = d(P, \ell)$.

Theorem 8.3.3. *If* ℓ *and* m *are parallel lines and there exist two points on* m *that are equidistant from* ℓ, *then* ℓ *and* m *admit a common perpendicular.*

Theorem 8.3.4. *If lines* ℓ *and* m *admit a common perpendicular, then that common perpendicular is unique.*