Theorem 5.7.15 (The Crossbar Theorem): *If $\triangle ABC$ is a triangle and D is a point in the interior of $\angle BAC$, then there is a point G such that G lies on both $\overrightarrow{AD}$ and $\overline{BC}$.*

Proof: We shall prove the Crossbar Theorem by contradiction. To begin, let $\triangle ABC$ be a triangle and let D be a point in the interior of $\angle BAC$. Having that D is an interior point of $\angle BAC$, we know that the points D and C are on the same side of $\overleftrightarrow{AB}$ and the points D and B are on the same side of $\overleftrightarrow{AC}$ by the definition for the interior of an angle. Additionally, the definition of betweenness for rays tells us that the ray $\overrightarrow{AD}$ is between the rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

Using the fourth part of the Protractor Postulate, we may state that since the ray $\overrightarrow{AD}$ is between the rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$, then $\mu\angle BAD + \mu\angle DAC = \mu\angle BAC$.

Now, let us assume that the points B and C are on the same side of $\overleftrightarrow{AD}$. This will be our RAA hypothesis. Using this assumption with the fact that the points D and C are on the same side of $\overleftrightarrow{AB}$, we may use the definition of betweenness for rays to safely say that $\overrightarrow{AC}$ is between $\overrightarrow{AB}$ and $\overrightarrow{AD}$. Likewise, we may use the assumption with the fact that the points D and B are on the same side of $\overleftrightarrow{AC}$ to safely say that $\overrightarrow{AB}$ is between $\overrightarrow{AC}$ and $\overrightarrow{AD}$. Thus we have that each ray is between the other two rays.

Using the fourth part of the Protractor Postulate with the fact that $\overrightarrow{AB}$ is between $\overrightarrow{AC}$ and $\overrightarrow{AD}$, we get that $\mu\angle CAB + \mu\angle BAD = \mu\angle CAD$. Because $\angle CAD = \angle DAC$ and $\angle CAB = \angle BAC$, we may exchange these terms in this equation and then substitute the equation in for $\mu\angle BAC$ in our earlier equation getting us $\mu\angle BAD + (\mu\angle BAC + \mu\angle BAD) = \mu\angle BAC$. Simplifying, we get that $\mu\angle BAC + 2\mu\angle BAD = \mu\angle BAC$. The Protractor Postulate defines that these measures are real numbers, so we may use the rules of arithmetic to get that $2\mu\angle BAD = 0°$ and consequently that $\mu\angle BAD = 0°$. The second part of the Protractor Postulate affirms that $\mu\angle BAD = 0°$ if and only if $\overrightarrow{AB} = \overrightarrow{AD}$ and as such A, B and D are collinear.

However a problem arises when we realize that since D lies on the ray $\overrightarrow{AB}$, it cannot be on the same side of $\overleftrightarrow{AB}$ as the point C and therefore could not be an interior point of $\angle BAC$ but this would contradict the primary hypothesis that D is in fact an interior point. Thus our assumption is false and the points B and C must therefore lie on opposite sides of $\overleftrightarrow{AD}$.

The second part of the Plane Separation Postulate tells us that since the points B and C are on opposite sides of $\overleftrightarrow{AD}$ then the line segment $\overline{BC}$ intersects the line $\overleftrightarrow{AD}$. Let us call this point of intersection G. Now we know that G lies on $\overleftrightarrow{AD}$ and $\overline{BC}$ but we need to check to be certain that G lies on the ray $\overrightarrow{AD}$ and to do this we will use another contradiction.

First let us note that since G is a point on $\overline{BC}$ it is an interior point of $\angle BAC$. This is clear because G is between B and C and since there is only one point of intersection for every pair of lines according to a previously proven theorem, $\overline{GC}$ does not intersect $\overrightarrow{AB}$ and $\overline{BG}$ does not intersect $\overrightarrow{AC}$ because their would be points of intersection are B and C respectively, and those points do not lie in $\overline{GC}$ and $\overline{BG}$ respectively.

Now, let us assume that the point G does not lie on the ray $\overrightarrow{AD}$, then since it lies on the line $\overleftrightarrow{AD}$, it must lie on the opposite ray $\overrightarrow{AG}$ because this is merely an application of set theory since G belongs to the set $\overleftrightarrow{AD}$ minus the set $\overrightarrow{AD}$. Observing now that $\overrightarrow{AD}$ and $\overrightarrow{AG}$ are opposite rays, we see that A must be between G and D otherwise either G would lie on the ray $\overrightarrow{AD}$ or D would lie on the ray $\overrightarrow{AG}$.

Additionally, we have the important fact that $\overleftrightarrow{GD}$ intersects $\overleftrightarrow{AB}$ and $\overleftrightarrow{GD}$ intersects $\overleftrightarrow{AC}$, and in both cases this intersection occurs at the point A and we know that it occurs only at this point, again by a previously proven theorem. Because the lines intersect and the point A is between G and D, we know that the line segment $\overline{GD}$ intersects both $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$. Thus, we may say that the points G and D lie on opposite sides of both the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$ by the Plane Separation Postulate and the definition for points lying on opposite sides of a line.

On the other hand, returning back to the fact that D and G are interior points of $\angle BAC$, we know that they are both on the same side of $\overleftrightarrow{AB}$ as the point C and they are both on the same side of $\overleftrightarrow{AC}$ as the point B and thus they must both be on the same side of $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$ as one another, but this contradicts what we just found, so our assumption must be false and G must lie on the ray $\overrightarrow{AD}$. Q.E.D.