Math 443 **Differential Geometry** **Spring 2011**

Assignment #9 (CORRECTED) (not to be handed in)

A. Using isometries or otherwise, prove the following ***AAA Congruence Theorem*** on the hyperbolic plane: *If $\triangle ABC$ and $\triangle A'B'C'$ are two geodesic triangles in $\mathbb{M}_{-1}$ such that interior angle measures at A, B, and C are equal to the measures at A', B', and C', respectively, then $\triangle ABC$ is congruent to $\triangle A'B'C'$.*

B. Suppose $S \subset \mathbb{R}^3$ is a regular surface with a Riemannian metric g, and suppose $R \subset S$ is a regular region diffeomorphic to the cylinder $S^1 \times [0,1]$, whose boundary curves are both simple closed geodesics. If the Gauss curvature in R is not identically zero, prove that it attains both positive and negative values.

C. Suppose $S \subset \mathbb{R}^3$ is a compact orientable regular surface that is diffeomorphic to a torus, endowed with the first fundamental form. Prove that S contains points p_1, p_2, p_3 such that $K(p_1) > 0$, $K(p_2) = 0$, and $K(p_2) < 0$.

D. Compute the Euler characteristic of each of the following regular surfaces in $\mathbb{R}^3$:

(a) An ellipsoid defined by
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

(b) The surface S defined by
$$x^2 + y^{10} + z^6 = 1.$$

[Hint: you don't need to construct a triangulation.]

E. A ***closed geodesic*** on a surface is a geodesic with a periodic parametrization. For each of the following Gauss curvature conditions, determine whether there can exist a closed geodesic on a surface satisfying the condition. In each case, either produce an example or prove that one cannot exist. (Note that this is not the same as asking whether there exists a geodesic 0-gon. Why?)

(a) $K > 0$.

(b) $K = 0$.

(c) $K < 0$.

F. Let $T \subset \mathbb{R}^3$ be the torus of revolution obtained by revolving the curve $(r-2)^2 + z^2 = 1$ about the z-axis (see Problem F on Assignment 4 from Math 442), endowed with the first fundamental form. Verify the Gauss–Bonnet theorem for T by computing each term separately.

G. Let S be the paraboloid defined by $z = x^2 + y^2$, and for each $r > 0$, let S_r be the portion of S where $z \le r$. Verify the Gauss–Bonnet formula for S_r by computing each term separately.

H. Let φ and λ be latitude and longitude on the unit sphere (see Bär, p. 211), and let R be the region defined by $0 \le \varphi \le \pi/4$. Verify the Gauss–Bonnet formula for R by computing each term separately.