Math 443 **Differential Geometry** **Spring 2011**

Assignment #8 (2ND CORRECTION): Due 5/27/11

Reading:

- [Optional] Do Carmo, Section 4.5 (pp. 264–282).

Written Assignment:

A. Suppose S is a regular surface with a Riemannian metric g. A ***geodesic polygon*** in S is a simple region whose edges are geodesic segments, and with no cusp vertices. It is called a ***geodesic n-gon*** if it has exactly n ordinary vertices.

   (a) Show that if S has nonpositive Gauss curvature, then S does not contain any geodesic n-gons with $0 \le n \le 2$.

   (b) Now suppose S is the unit sphere with the first fundamental form. For which values of $n < 3$ do there exist geodesic n-gons in S? Prove your answer correct.

B. Suppose S is a regular surface with a Riemannian metric g. A ***geodesic triangle*** in S is a geodesic 3-gon (see Problem A). The ***angle excess*** of a geodesic triangle T is the quantity

$$E(T) = (\theta_1 + \theta_2 + \theta_3) - \pi,$$

where $\theta_1, \theta_2, \theta_3$ are the interior angle measures of T. Prove that every geodesic triangle satisfies $E(T) = \int_T K\,dA$.

C. Let $(\mathbb{M}_{-1}, g)$ be the hyperbolic plane. Prove that the area of every geodesic triangle in $\mathbb{M}_{-1}$ is strictly less than π.

D. Two geodesic triangles in a surface S are said to be ***congruent*** if there is a correspondence between their vertices such that corresponding side lengths are equal and corresponding interior angle measures are equal. Let $(\mathbb{M}_{-1}, g)$ be the hyperbolic plane. Prove that geodesic triangles in $\mathbb{M}_{-1}$ satisfy the ***SAS congruence theorem:*** *If $\triangle ABC$ and $\triangle A'B'C'$ are two geodesic triangles in $\mathbb{M}_{-1}$ such that $L_g(\overline{AB}) = L_g(\overline{A'B'})$, $L_g(\overline{AC}) = L_g(\overline{A'C'})$, and the interior angle measures at A and A' are equal, then $\triangle ABC$ is congruent to $\triangle A'B'C'$.* [Hint: use isometries.]

E. Suppose S is an oriented surface with a Riemannian metric g and constant Gauss curvature k. For any $r > 0$, let $D_r(0)$ and $S_r(0)$ denote the closed disk and circle of radius r centered at the origin in $\mathbb{R}^2$:

$$D_r(0) = \{(u,v) : u^2 + v^2 \le r^2\}, \qquad S_r(0) = \{(u,v) : u^2 + v^2 = r^2\}.$$

Let $p \in S$, and suppose $F\colon U \to S$ is a Riemannian normal coordinate parametrization. For any $r > 0$ such that $D_r(0) \subset U$, the ***geodesic disk of radius r centered at p*** is the set $D_r(p) = F(D_r(0)) \subset S$, and the ***geodesic circle of radius r centered at p*** is $S_r(p) = F(S_r(0)) \subset S$. Compute the area of $D_r(p)$, the circumference of $S_r(p)$, and the geodesic curvature of $S_r(p)$ in terms of k and r; and verify that they satisfy the Gauss–Bonnet formula. [Hint: use geodesic polar coordinates.]