Math 443 **Differential Geometry** **Spring 2011**

Assignment #5: Due 5/2/11

MIDTERM EXAM: Wednesday, May 4, in class.

Written Assignment (not to be handed in for a grade):

A. Exercise 4.32 (page 209).

B. Exercise 4.33 (page 209).

C. Let S^2 denote the unit sphere in $\mathbb{R}^3$, with its first fundamental form. Consider the following map $F\colon \mathbb{R}^2 \to \mathbb{R}^3$:

$$F(\varphi, x) = (\operatorname{sech} x \cos\varphi, \operatorname{sech} x \sin\varphi, \tanh x).$$

(a) Show that if (φ, x) is restricted a a suitably small open set in the plane, F is a local parametrization of S^2.

(b) Show that meridians and latitude lines on the sphere are images under F of vertical and horizontal straight lines in $\mathbb{R}^2$.

(c) Show that the image of a nonvertical and nonhorizontal straight line in $\mathbb{R}^2$ is a curve that makes a constant angle with latitude lines.

D. A point on a surface (with the first fundamental form) is said to be ***umbilic*** if the principal curvatures are equal at that point. Find all umbilic points on the ellipsoid given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

where a, b, c are positive constants.

E. Let $S \subset \mathbb{R}^3$ be the surface $\{(x, y, z) : y > 0, z = 0\}$ (the positive-y portion of the xy-plane), with the global parametrization $F(u, v) = (u, v, 0)$ for $u \in \mathbb{R}$ and $v > 0$. Let g be the Riemannian metric on S with the following matrix in this parametrization:

$$(g_{ij}) = \begin{pmatrix} \dfrac{1}{v^2} & 0 \\ 0 & \dfrac{1}{v^2} \end{pmatrix}.$$

Let $\mathbb{M}_{-1}$ be the hyperbolic plane, and define a map $F\colon S \to \mathbb{M}_{-1}$ by

$$F(x, y, 0) = \left(\frac{x}{y}, \ \frac{x^2 + y^2 - 1}{2y}, \ \frac{x^2 + y^2 + 1}{2y} \right).$$

Prove that F is a global isometry.

F. Compute the Gauss curvature of the surface S of the preceding problem. [Hint: you shouldn't have to do any complicated computations.]