Math 442 **Differential Geometry** **Winter 2013**

Assignment #2
Supplementary Exercises (CORRECTED 1/20/13)

S4. Let $\alpha, \beta\colon I \to \mathbb{R}^n$ be smooth functions. Prove that

$$\frac{d}{dt}\langle \alpha(t), \beta(t)\rangle = \langle \alpha'(t), \beta(t)\rangle + \langle \alpha(t), \beta'(t)\rangle.$$

S5. Let A be an 3×3 matrix. Then A is said to be ***orthogonal*** if $A^T A$ is equal to the identity matrix (where A^T denotes the transpose of A). Show that the following are equivalent.
(a) A is an orthogonal matrix.
(b) The columns of A are orthonormal.
(c) $\langle Av, Aw\rangle = \langle v, w\rangle$ for all $v, w \in \mathbb{R}^3$.
[Hint: It might be useful to show that $\langle v, w\rangle = v^T w$ for vectors v, w, thought of as column matrices, and that $(Av)^T = v^T A^T$.]

S6. Find an explicit unit-speed parametrization of the curve $\sigma\colon (0,\infty) \to \mathbb{R}^2$ given by $\sigma(t) = (t^2, t^3)$. Be sure to specify the domain of the new parametrization.

S7. Let $\sigma\colon I \to \mathbb{R}^3$ be a smooth, unit-speed, biregular curve, and $s_0 \in I$. Show that there is a unique unit-speed parametrized circle $\gamma\colon \mathbb{R} \to \mathbb{R}^3$, called the ***osculating circle***, such that

$$\gamma(s_0) = \sigma(s_0),$$
$$\gamma'(s_0) = \sigma'(s_0),$$
$$\gamma''(s_0) = \sigma''(s_0).$$

Show that the radius of the osculating circle is $1/\kappa(s_0)$. (You may use the fact that every unit-speed parametrized circle of radius r can be written in the form

$$\gamma(s) = \mathbf{p_0} + \left(r\cos\frac{s}{r}\right)\mathbf{v} + \left(r\sin\frac{s}{r}\right)\mathbf{w}$$

for some $\mathbf{p_0} \in \mathbb{R}^3$ and some pair of orthogonal unit vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$.)