BACKGROUND MATERIAL ON LINEAR ALGEBRA AND REAL ANALYSIS

References

(These are on reserve in Odegaard and in the Math Research Library, Padelford C-306.)

[S] J. Stewart, *Calculus: Early Transcendentals, 4th ed.*, Brooks/Cole, 1999

[JRA] L. W. Johnson, R. D. Riess, and J. T. Arnold, *Introduction to Linear Algebra, 5th ed.*, Addison-Wesley, 2002

[TM] A. E. Taylor and W. R. Mann, *Advanced Calculus, 3rd ed.*, Wiley, 1983

Euclidean Spaces

Throughout this course, $\mathbb{R}$ will denote the set of real numbers, and for any positive integer n, $\mathbb{R}^n$ will denote the set of ordered n-tuples of real numbers. We will be working almost exclusively with $\mathbb{R}^1 = \mathbb{R}$, $\mathbb{R}^2$, and $\mathbb{R}^3$, which are called the line, the Euclidean plane, and (3-dimensional) Euclidean space, respectively. In these cases, an element of $\mathbb{R}^n$ can be visualized in two ways: On the one hand, we can think of an element $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ as a *point* in $\mathbb{R}^n$, distinguished only by its location; in this case, the numbers $x_1, \dots, x_n$ are called the *coordinates* of the point. On the other hand, we can think of x as a *vector*, visualized as an arrow and distinguished only by its direction and length, irrespective of where its starting point may be placed; in this case, $x_1, \dots, x_n$ are called the *components* of the vector. The relationship between the two pictures is the following: If x is the point whose coordinates are $(x_1, \dots, x_n)$, then the vector whose components are $(x_1, \dots, x_n)$ is represented by the arrow from the origin to x. More generally, given two points $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$, the vector starting at x and ending at y, denoted by $y - x$, is the vector whose components are $(y_1 - x_1, \dots, y_n - x_n)$.

Although we will use points and vectors for different purposes, and will talk about them as if they were different things, both are represented mathematically as ordered n-tuples of real numbers. Real numbers are frequently called *scalars* to distinguish them from vectors or points.

If $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ are vectors in $\mathbb{R}^n$, their *dot product* (also called their *scalar product* or *inner product*) is the scalar value defined by

$$x \cdot y = x_1 y_1 + \cdots + x_n y_n.$$

The *length* or *norm* of a vector $x \in \mathbb{R}^n$, denoted by $|x|$, is defined by

$$|x| = \sqrt{x \cdot x} = \sqrt{(x_1)^2 + \cdots + (x_n)^2}.$$

(Many authors use the notation $\|x\|$ for the norm of x, but we will follow do Carmo and use the same symbol for both absolute values and norms.) The *distance* between two points $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ in $\mathbb{R}^n$ is the length of the vector between them, namely

$$|x - y| = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}.$$

The geometric interpretation of the dot product is descibed by the following formula:

$$x \cdot y = |x|\,|y| \cos\theta,$$

where θ is the angle between the vectors x and y, which is uniquely determined as long as we require that $0 \le \theta \le \pi$.

Linear Algebra

A *vector space* is a set V, whose elements are usually called *vectors*, on which two operations are defined:

- *vector addition*, which takes two vectors $x, y \in V$ and yields a vector $x + y$; and
- *scalar multiplication*, which takes a vector $x \in V$ and a real number a and yields a vector ax.

These operations are required to satisfy the following properties for all $x, y, z \in V$ and all $a, b \in \mathbb{R}$:

(a) $x + y$ is an element of V.

(b) ax is an element of V.

(c) $x + y = y + x$.

(d) $x + (y + z) = (x + y) + z$.

(e) There exists a vector $0 \in V$ (sometimes denoted by $\boldsymbol{\theta}$) with the property that $0 + x = x$ for every $x \in V$.

(f) For each $x \in V$, there exists a vector $-x \in V$ such that $x + (-x) = 0$.

(g) $a(bx) = (ab)x$.

(h) $a(x + y) = ax + ay$.

(i) $(a + b)x = ax + bx$.

(j) For every vector $x \in V$, $1x = x$.

If W is a vector space, a subset $V \subset W$ is called a *subspace* (or sometimes a *linear subspace* or a *vector subspace*) if it satisfies properties (a)–(j) above and is therefore a vector space in its own right. It is not hard to show that $V \subset W$ is a subspace if and only if $0 \in V$ and V is closed under addition and scalar multiplication (properties (a) and (b)).

The most important example of a vector space is, of course, $\mathbb{R}^n$, with vector addition and scalar multiplication defined by

$$\begin{aligned}(x_1, \dots, x_n) + (y_1, \dots, y_n) &= (x_1 + y_1, \dots, x_n + y_n),\\ a(x_1, \dots, x_n) &= (ax_1, \dots, ax_n).\end{aligned}$$

The only other vector spaces we will be considering in this course are subspaces of some $\mathbb{R}^n$ (typically $\mathbb{R}^2$ or $\mathbb{R}^3$), so you are free to interpret the term "vector space" as being synonymous with "subspace of $\mathbb{R}^n$."

If V and W are vector spaces, a function $T\colon V \to W$ is said to be a *linear map* (or *linear transformation*) if it satisfies the following properties: For all $x, y \in V$ and all $a \in \mathbb{R}$,

$$\begin{aligned}T(x + y) &= T(x) + T(y),\\ T(ax) &= aT(x).\end{aligned}$$

Make sure you are familiar with the following concepts (references in brackets refer to chapters in the books listed at the beginning of this handout).

- Cross products of vectors in $\mathbb{R}^3$ [S, C12; JRA, C2].
- Equations for lines and planes in $\mathbb{R}^2$ and $\mathbb{R}^3$ [S, C12; JRA, C2].

- Matrix operations (multiplication by scalars, matrix addition and multiplication) [JRA, C1].
- The transpose of a matrix; symmetric matrices [JRA, C1].
- Singular and nonsingular matrices [JRA, C1].
- The inverse of a nonsingular matrix; the identity matrix [JRA, C1].
- Linear dependence and independence [JRA, C1].
- The span of a set of vectors [JRA, C3].
- Bases and dimension of a vector space [JRA, C3].
- The matrix of a linear map [JRA, C3].
- The range, nullspace, rank, and nullity of a matrix or a linear map [JRA, C3].
- Orthogonal and orthonormal bases [JRA, C3].
- Eigenvalues, eigenvectors, and eigenspaces of a linear map [JRA, C4].
- Determinants of 2×2 and 3×3 matrices [JRA, C4].

Real Analysis

If x is a point in $\mathbb{R}^n$ and ε is a positive real number, the *(open) ball* with center x and radius ε is the set

$$B_\varepsilon(x) = \{y \in \mathbb{R}^n : |y - x| < \varepsilon\}.$$

A subset $U \subset \mathbb{R}^n$ is said to be *open* in $\mathbb{R}^n$ if for every $x \in U$, there exists some $\varepsilon > 0$ such that $B_\varepsilon(x) \subset U$. If $x \in \mathbb{R}^n$, a *neighborhood of x in $\mathbb{R}^n$* is an open subset of $\mathbb{R}^n$ containing x.

Suppose $U \subset \mathbb{R}^n$ is an open set and $F\colon U \to \mathbb{R}^m$ is any map. If x_0 is a point of U, F is said to be *continuous at x_0* if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $x \in U$ and $|x - x_0| < \delta$ imply $|F(x) - F(x_0)| < \varepsilon$. If F is continuous at every point of U, we say simply that F is continuous in U.

We can also define continuity for functions whose domains are arbitrary (not necessarily open) subsets of $\mathbb{R}^n$. Let $A \subset \mathbb{R}^n$ be any subset, and let $F\colon A \to \mathbb{R}^m$ be a map. The usual mathematical definition of continuity of F is a straightforward adaptation of the definition on open sets: $F\colon A \to \mathbb{R}^m$ is continuous if for every $x_0 \in A$ and every $\varepsilon > 0$, there exists $\delta > 0$ such that $x \in A$ and $|x - x_0| < \delta$ imply $|F(x) - F(x_0)| < \varepsilon$. However, do Carmo uses a slightly different definition: He says that $F\colon A \to \mathbb{R}^m$ is continuous if there exists an open set $U \subset \mathbb{R}^m$ containing A and a continuous map $\widetilde{F}\colon U \to \mathbb{R}^m$ whose restriction to A is equal to F. These definitions are *not* equivalent (see below), but do Carmo's definition will suffice for our purposes so we will use it.

It is easy to check that any function that is continuous by do Carmo's definition is also continuous by the ε-δ definition. But, for those of you who are interested, here is an example that shows that the converse is not always true. Let $A \subset \mathbb{R}^2$ be the subset

$$A = \{(x, y) \in \mathbb{R}^2 : y \neq 0\} \cup \{(0, 0)\},$$

and define $F\colon A \to \mathbb{R}$ by

$$F(x,y) = \begin{cases} x, & y > 0, \\ 0, & (x,y) = (0,0), \\ -x, & y < 0. \end{cases}$$

Then F is continuous by the ε-δ definition, but it is not continuous by do Carmo's definition.

Let $U \subset \mathbb{R}^n$ be an open set and let $f\colon U \to \mathbb{R}$ be a map. If $x_0 = (x_1^0, \dots, x_n^0)$ is a point of U and $1 \le i \le n$, the *ith partial derivative of f at x_0* is defined, as usual, as the ordinary derivative of f with respect to x_i at $x_i = x_i^0$, with all the other variables held fixed:

$$\frac{\partial f}{\partial x_i}(x_0) = \lim_{h\to 0} \frac{f(x_1^0, \dots, x_{i-1}^0, x_i^0 + h, x_{i+1}^0, \dots, x_n^0) - f(x_1^0, \dots, x_n^0)}{h},$$

if the limit exists. For a vector-valued map $F\colon \mathbb{R}^n \to \mathbb{R}^m$, we can write $F(x_1, \dots, x_n) = (F_1(x_1, \dots, x_n), \dots, F_m(x_1, \dots, x_n))$, where $F_1, \dots, F_m$ are the *component functions* of F. The partial derivatives of F at x_0 are just the partial derivatives of its component functions, denoted by $\partial F_i/\partial x_j(x_0)$. If these partial derivatives exist at each point $x_0 \in U$, we obtain mn new functions $\partial F_i/\partial x_j\colon U \to \mathbb{R}$, and we can ask in turn whether their partial derivatives exist. If they do, they are called the second partial derivatives of F, denoted by $\partial^2 F_i/\partial x_j \partial x_k$. Continuing in this way, the $(k+1)$st partial derivatives of F are defined to be the (first) partial derivatives of the kth partial derivatives. We say that F is *smooth* (or *infinitely differentiable*, or *of class C^∞*) if its partial derivatives of *all* orders exist and are continuous on U.

In do Carmo's book, the term *differentiable* is used as a synonym for smooth. This is non-standard terminology; for example, most authors define a differentiable function $f\colon \mathbb{R} \to \mathbb{R}$ simply as one that possesses a derivative at each point. Because of the potential for confusion, I will generally avoid using the term "differentiable," and call such functions "smooth" instead; but when you read do Carmo, you need to be aware that differentiable for him means infinitely differentiable.

If $U \subset \mathbb{R}^n$ is open, $x \in U$, and $F\colon U \to \mathbb{R}^m$ is a differentiable map, the *Jacobian matrix* of F at x is the $m \times n$ matrix of partial derivatives of the component functions of F at x:

$$\begin{pmatrix} \dfrac{\partial F_1}{\partial x_1}(x) & \dots & \dfrac{\partial F_1}{\partial x_n}(x) \\ \vdots & \ddots & \vdots \\ \dfrac{\partial F_m}{\partial x_1}(x) & \dots & \dfrac{\partial F_m}{\partial x_n}(x) \end{pmatrix}.$$

If $m = n$ (so that the Jacobian matrix is square), the *Jacobian determinant* of F is the determinant of the Jacobian matrix, usually denoted by the following notation:

$$\frac{\partial(F_1, \dots, F_n)}{\partial(x_1, \dots, x_n)}(x) = \begin{vmatrix} \dfrac{\partial F_1}{\partial x_1}(x) & \dots & \dfrac{\partial F_1}{\partial x_n}(x) \\ \vdots & \ddots & \vdots \\ \dfrac{\partial F_n}{\partial x_1}(x) & \dots & \dfrac{\partial F_n}{\partial x_n}(x) \end{vmatrix}.$$

Make sure you are familiar with the following concepts from elementary real analysis. References in brackets refer to chapters in the books listed at the beginning of this handout.

- Open and closed sets, limit points, and boundary points in $\mathbb{R}$ and $\mathbb{R}^2$ [TM, C5].
- The chain rule for functions of two and three variables [TM, C6 & C7]
- The inverse and implicit function theorems [TM, C8 & C9].
- Connected subsets of $\mathbb{R}^2$ and $\mathbb{R}^3$ [TM, C7 & C17].
- The intermediate value theorem [TM, C3 & C17].
- The Heine-Borel theorem [TM, C16].
- The extreme value theorem [TM, C17].
- Transformation of double integrals under a change of variables [TM, C15].