Math 441 **Topology** **Summer 2007**

Handout #5: Conventions for Writing Mathematical Proofs

Here are some of the standard conventions for mathematical proof-writing.

- **Write in paragraph form:** First and foremost, remember always that a mathematical proof is designed to communicate the truth of a mathematical statement to a *human reader*. Ordinary prose is almost always better suited to this purpose than formal symbolic statements. Although you will probably construct your proof as a sequence of terse symbolic statements, when you write it up you must use complete sentences organized into paragraphs. As you read more and more complicated proofs, you will find that paragraph-style proofs are much easier to read and comprehend than symbolic ones or the two-column proofs of high school geometry.

- **Use proper English:** All mathematical writing should follow the same conventions of grammar, usage, punctuation, and spelling as any other writing. In addition to writing complete sentences organized into paragraphs, you must use correct punctuation (including a period at the end of every sentence), avoid run-on sentences, pay attention to subject-verb agreement and parallel structure, and use correct spelling and capitalization. If you are not a native English speaker, it might be a good idea to ask a native speaker to look over your first few proofs before you submit them.

- **Identify your audience:** Before you begin writing any proof, be sure you're aware who your audience is and what they already know. For this course, you should always write your proofs as if you were trying to convince a fellow student in this class of the truth of the theorem and the correctness of your argument. You may assume the reader knows the same background material as you do, but doesn't know the proof of this particular theorem.

- **Include the right amount of detail:** A clear awareness of your audience will help you to answer the perennial question, "How much detail do I need to include?" The first thing that must be said is this: *If you think you probably know roughly how an argument would go but it seems too tedious to work through in detail, then you need to work through it!* It's only after you know exactly what's involved in writing out the details that you can make a good judgment about whether those details need to be included in the proof or not. If you're sure that it would be obvious to your audience how to fill in the omitted details, then the proof might be clearer if you leave them out. But if they weren't obvious to you at first, then something probably needs to be said—it might not be necessary to write down every step, but you should include just enough to give the reader the "Aha!" experience that makes the rest obvious (and, just as importantly, makes it clear to the grader that you've figured out the details yourself!). Deciding how much detail to include is one of the most subtle and difficult aspects of writing, and one where experience and artistry are most evident.

- **Label your theorems clearly:** It isn't necessary to copy out the problem statements verbatim. However, each theorem you prove should be clearly labeled as such, and stated clearly and precisely in one or more English sentences. With computer typesetting programs, the usual convention is to set the word "Theorem" in boldface, with the statement of the theorem itself italicized. In handwritten proofs, just underline the word "Theorem."

  In some contexts, the word Theorem might be replaced by Proposition, Corollary, or Lemma. Logically, these all mean the same thing (a mathematical statement to be proved from the axioms and previously proved theorems), but your choice of label can alert the reader about the role that the result plays in the current context. A *proposition* is a result that is interesting in its own right, but not as important as a theorem; a *lemma* is a result that might not be interesting in itself, but is useful for proving another theorem; and a *corollary* is a result that follows easily from a previously proved theorem.

- **Show where your proofs begin and end.** Each proof should begin with the word *Proof*, and end either with the letters QED (*quod erat demonstrandum*, Latin for "that which was to be proved") or with a symbol such as the square at the end of this paragraph. $\square$

- **Write with precision:** In mathematical writing more than any other kind, precision is of paramount importance. Every mathematical statement you make must have a precise mathematical meaning; every term you use must have a precise mathematical definition and be used correctly according to its definition. The only place for statements that are vague or intuitive is in a discussion that is clearly identified as not being part of the formal proof.

- **Define your terms:** If you use a mathematical term that your audience might not know the meaning of, include its definition. Whenever you use a symbolic variable, make sure that you give it a precise mathematical meaning *before* the first time you refer to it. There are several ways to give meaning to a variable:

  1. By quantifying it in an appropriate way: "For every $x \in \mathbb{R}$, ..."
  2. By assuming it is an arbitrary element of some set: "Let $x \in \mathbb{R}$."
  3. By setting it equal to some specific, well-defined mathematical object: "Let $x = y + z$" or "Let x be the unique positive number such that $x^2 = y$." The words "set" and "define" are also used often in this context. If you use an equation to define a variable, the variable being defined must be on the *left-hand side* of the equation. Thus "Let $y + z = x$" is not an acceptable way to define the variable x.

- **Remember that every step needs justification:** Each step of a proof must be clearly justified in one of six ways: By definition, by an axiom, by a previous theorem, by hypothesis, by a previous step in the same proof, or by the rules of logic. When you first start writing proofs, you should generally state the justification for each step. As you acquire more practice, you'll begin to see that the justifications of some statements will be so obvious to your readers that they don't need to be stated; but you should always make sure that *you* are aware of why each step is justified.

- **Ask Yourself the Key Questions:** You should be prepared to answer these two questions about each sentence you write:

  – *What exactly does this mean?*
  – *Why exactly is this true?*

- **Write clearly:** Just as important as mathematical precision is making sure your writing is clear enough to be easily comprehensible to your intended audience. Don't be stingy with intuitive explanations of what's going on and why. For any argument that's longer than a few sentences, it's good to begin by describing informally what you're going to do, then do it, then say what you've done. If the structure of your proof is anything other than a simple direct proof, state at the beginning what type of proof you're using. ("We will prove the contrapositive," or "We will prove this by contradiction.")

- **Intersperse formulas with explanatory text:** It's all too easy to write a sequence of symbolic mathematical statements that are entirely precise and mathematically correct, and yet that are incomprehensible to a human being. If you have to write a long series of equations or inequalities, intersperse them at carefully chosen places with some words about of what you're doing and why, or reasons why one step follows from another.

- **Distinguish formal vs. informal mathematical writing:** Many mathematical papers include both formal and informal parts. The *formal* part lays out the precise mathematical definitions and describes the logical steps of the proof. The *informal* part might describe the motivation for why the theorem should be true, or the intuition behind the proof, or a brief sketch of how the proof will go. Be sure it is easy for the reader to distinguish which parts are formal and which are informal.

- **Use the first person singular sparingly:** Most authors avoid using the word "I" in mathematical writing. It is standard practice to use "we" whenever it can reasonably be interpreted as referring to "the writer and the reader." Thus: "We will prove the theorem by induction on n," and "Because f is injective, we see that $x_1 = x_2$." But if you're really referring only to yourself, it's better to go ahead and use "I": "I learned this technique from Richard Melrose."

- **Avoid most abbreviations:** There are a host of abbreviations that we use frequently in informal mathematical communication: "iff" (if and only if), "s.t." (such that), and "w.r.t." (with respect to) are some of the most common. These are indispensable for writing on the blackboard and taking notes, but should never be used in written mathematical exposition. The only exceptions are abbreviations that would be acceptable in any formal writing, such as "i.e." (that is) or "e.g." (for example); but if you use these, be sure you know the difference between them!

- **Proofread:** Be sure to read your proofs from beginning to end after you've finished writing them. You'll be amazed how many silly mistakes you can catch that way.

Mathematical formulas

The feature that most clearly distinguishes mathematical writing from other kinds is the extensive use of symbols and formulas. Here are some guidelines for using mathematical symbols in your writing. In this handout, the use of the word "symbol" includes letters such as x, y, A, B, α, β used to represent mathematical objects; function names such as f, sin, log; as well as all the special mathematical symbols that we use to refer to operators, relations, and other such mathematical constructions, such as $+$, $=$, $\cap$, or $\perp$. The word "formula" refers to any expression built up out of one or more symbols, such as $x + y$, $A \cap B$, $\triangle ABC$, or $f(x) = x^2 - 2$.

- *Mathematical* symbols and formulas are your friends! There's nothing wrong with including them in your proofs—if used appropriately, they can make mathematical writing dramatically easier to read. The sentence "The area is one-half of the product of the altitude multiplied by the base" is far less clear than "The area is $\frac{1}{2}bh$." On the other hand, symbols must be used judiciously, because their excessive use can lead to writing that is just as obscure as writing with no symbols.

- *Logical* symbols, such as $\exists$ (there exists), $\forall$ (for all), $\wedge$ (and), $\vee$ (or), $\neg$ (not), $\Rightarrow$ (implies), or $\Leftrightarrow$ (if and only if), should *never* be used to replace the corresponding words in an English sentence. The only time most of these symbols have any place in formal mathematical writing is as part of complete symbolic logic formulas; but even then, unless the subject you are writing about is mathematical logic, it is usually clearer to write out the statements in English. The only logical symbols that are commonly used in ordinary mathematical writing are $\Rightarrow$ and $\Leftrightarrow$, and then only to connect symbolic statements (or letters representing statements), as in "$x \neq 0 \Rightarrow x^2 > 0$" or "we will prove that (a) $\Leftrightarrow$ (b) by first showing that (a) $\Rightarrow$ (b) and then showing that (b) $\Rightarrow$ (a)."

- Single symbols and short, simple formulas should usually be included right in your paragraphs without breaking the flow; but a formula that is large or especially important should be centered on a line by itself. (This is called a *displayed formula.*)

- If a displayed formula ends a sentence, it must be followed by a period.

- That last one is easy to forget, so let me say it again with emphasis: *If a displayed formula ends a sentence, it must be followed by a period.*

- Every mathematical symbol or formula, whether included in the text or displayed, must have a definite grammatical function in a sentence. Formulas should almost always have one of the following two grammatical functions: (1) An expression representing a particular mathematical object can be used as a noun; and (2) a complete symbolic mathematical statement can be used as a clause. For example, consider the following sentence:

> If $x > 2$, we see that $x^2 + x$ must be greater than 6.

Here the formula "$x > 2$" is a mathematical statement functioning as a clause (whose verb is "$>$"), while "$x^2 + x$" and "6" are mathematical expressions (representing real numbers) that function as nouns.

The best way to ensure that your formulas function grammatically correctly is to read each sentence aloud. When you do so, bear in mind that many symbols can be read in several different ways—for example, the symbol "=" can be read as "equals," "equal to," "be equal to," "is equal to," or "which is equal to," depending on context.

- Symbols representing mathematical relations (like $=$, $>$, $\in$, or $\subset$) or operators (like $+$, $-$, or $\cap$) should be used only to connect other mathematical symbols, not to connect words with symbols or with each other. For example:

  If x is a real number that is > 2, then $x^2 + x$ must be > 6. (WRONG)
  If x is a real number and $x > 2$, then we must have $x^2 + x > 6$. (RIGHT)

- Complicated expressions such as built-up fractions should be either displayed or written in such a way that they fit easily on a line without forcing extra spacing between lines. In particular, if a fraction or fractional expressions is included in the text, it should be written with a slash, as in "$x/(y + 2)$". If a fraction is so large or complicated that it needs to be written using a horizontal bar, it should be displayed. The only exception is small numerical fractions such as $\frac{1}{2}$, which can be included in text as long as they are written small enough to fit naturally on a line.

- It's bad form to begin a sentence with a mathematical symbol, because it makes it hard for the reader to recognize that a new sentence has begun. (You can't capitalize a symbol to indicate the beginning of a sentence!) It's usually easy to avoid this by minor rewording—for example, if you find yourself wanting to write a sentence that begins "f is a continuous function," you could write instead "The function f is continuous."

- Avoid writing two formulas separated only by a comma or other punctuation mark, because they will look like one long formula. For example, the sentence "If $x \neq 0$, $x^2 > 0$" can be confusing; it would be easier to read if a word were interposed between the two formulas, as in "If $x \neq 0$, then $x^2 > 0$."