Math 441 | **Topology** | **Summer 2007**

Assignment #7

Reading:

Munkres, §§51, 52(pp. 330–333), 52(pp. 335–337).

Weekly Report:

Due Sunday, August 5.

Written Problems:

Due Wednesday, August 8.

1. Suppose X is a compact topological space. Show that every infinite subset of X has a limit point in X. [Hint: If $A \subset X$ has no limit point, start by showing that every point of A has a neighborhood that contains at most one point of A.]

2. Suppose X is a compact topological space and $f\colon X \to \mathbb{R}$ is a continuous function that is everywhere positive: $f(x) > 0$ for all $x \in X$. Show that there is some positive number ε such that $f(x) \geq \varepsilon$ for all $x \in X$. Give a counterexample when X is not compact.

3. Show directly (without using the Heine–Borel theorem) that each of the following spaces is noncompact, by exhibiting an open cover with no finite subcover. Each space has the Euclidean topology. (Just give a precise definition of the cover; you don't have to prove that it has no finite subcover.)

   (a) $A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$.
   (b) $B = \{(x, y) \in \mathbb{R}^2 : y = x^2\}$.
   (c) $C = \{(x, y) \in \mathbb{R}^2 : 0 < x^2 + y^2 \leq 1\}$.
   (d) $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1 \text{ and } x \text{ is irrational}\}$.

4. Each of the following topological spaces is a set of 2×2 real matrices, which we think of as a subspace of $\mathbb{R}^4$ with the Euclidean topology. For each one, decide whether it is compact, and prove your answer correct.

   (a) $SL(2)$ is the set of 2×2 matrices with determinant equal to 1:

   $$SL(2) = \left\{ \begin{pmatrix} w & x \\ y & z \end{pmatrix} : wz - xy = 1 \right\}.$$

   (b) $O(2)$ is the set of 2×2 matrices whose rows are unit vectors that are orthogonal to each other:

   $$O(2) = \left\{ \begin{pmatrix} w & x \\ y & z \end{pmatrix} : w^2 + x^2 = 1,\ y^2 + z^2 = 1,\ wy + xz = 0 \right\}.$$

   (c) $SO(2)$ is the intersection of $SL(2)$ and $O(2)$:

   $$SO(2) = \left\{ \begin{pmatrix} w & x \\ y & z \end{pmatrix} : w^2 + x^2 = 1,\ y^2 + z^2 = 1,\ wy + xz = 0, \text{ and } wz - xy = 1 \right\}.$$

5. Optional Extra Credit: Decide whether each of the spaces in the preceding problem is connected, and prove your answer correct.

6. Optional Extra Credit: Let ∞ be some object not in $\mathbb{R}^2$, and let $(\mathbb{R}^2)^* = \mathbb{R}^2 \cup \{\infty\}$. (This is called the ***extended plane***.) Define a topology on $(\mathbb{R}^2)^*$ by declaring the open sets to be of the following two types:

   (i) Open subsets of $\mathbb{R}^2$ (in the usual topology);

   (ii) Sets of the form $\{\infty\} \cup (\mathbb{R}^2 \setminus K)$, where K is a compact subset of $\mathbb{R}^2$.

   (You may accept the fact that this is a topology.)

   (a) Prove that $(\mathbb{R}^2)^*$ is compact.

   (b) Define a map $F\colon (\mathbb{R}^2)^* \to S^2$ as follows:

$$F(x,y) = \left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1}\right), \qquad (x,y) \in \mathbb{R}^2,$$
$$F(\infty) = (0,0,1).$$

   Prove that F is a homeomorphism. [Hint: use the compact-Hausdorff lemma.]

Second Portfolio Assignment:

Now due Monday, August 6.

Write up careful and complete solutions to the following problems, with due attention to the conventions of mathematical proof-writing described in Handout #5:

- Munkres, §24, #10.
- Problem 5 on the midterm exam.

For this assignment, pretend that Munkres has decided that these problems belong in the textbook, instead of being left as problems, and has asked you to write the relevant section(s). Include a section title, some introductory explanation and motivation, any definitions that the reader will need, a precise theorem statement, and a clear proof. On Friday, August 3, bring two copies to class for peer review.