Math 324A & 324B Advanced Multivariable Calculus I **Spring 1997**

ANSWERS TO WORKSHEET #8

1. (a) $\vec{r}\,'(t) = (1, 6t^2, 6t)$ and $\overrightarrow{F}(\vec{r}(t)) = (0, t, 0)$, so

$$\int_C \overrightarrow{F} \cdot d\vec{r} = \int_0^1 (0, t, 0) \cdot (1, 6t^2, 6t) dt = \int_0^1 6t^3 \, dt = \frac{3}{2}.$$

   (b) This can be done directly by parametrizing the curve and substituting $x = t$, $y = 2t^3$, $z = 3t^2$, $dx = dt$, $dy = 6t^2\,dt$, and $dz = 6t\,dt$. But it is much easier to note that the vector field $(yz + z, xz, xy + x)$ is conservative by the curl test. A potential is $g(x, y, z) = xyz + xz$, so the line integral is $g(1, 2, 3) - g(0, 0, 0) = 9$.

2. First let's parametrize the line segment: $\vec{r}(t) = (t, t, t)$, for $0 \le t \le 3$. The velocity vector is $\vec{r}\,'(t) = (1, 1, 1)$, and the length of C is $\sqrt{3^2 + 3^2 + 3^2} = \sqrt{27}$.

   (a) The vector field is parallel to $\vec{r}\,'(t)$, so the tangential component is $|\overrightarrow{H}| = 1$, and the line integral is equal to $\sqrt{27}$.

   (b) The tangential component is $\overrightarrow{K} \cdot \vec{r}\,'(t)/|\vec{r}\,'(t)| = z^2/\sqrt{3}$. This isn't constant, but it takes its minimum value of 0 at the origin and its maximum value of $3^2/\sqrt{3}$ at $R(3, 3, 3)$. Therefore, the line integral is bounded between 0 and $3^2\sqrt{27}/\sqrt{3} = 81$.

   (c) Along D, $\overrightarrow{L} = (-t, t, 0)$ and $\overrightarrow{L} \cdot \vec{r}\,'(t) = -t + t = 0$, so the line integral is zero.

3. (a) This is a graph, so we use the graph parametrization:

$$\vec{r}(u, v) = \left(u, v, \frac{c(v - b)^2}{b^2}\right).$$

   (b) We need to compute $\overrightarrow{F}(\vec{r}(u, v)) = (c(v - b)^2/b^2, 0, v)$, $\vec{r}_u = (1, 0, 0)$, $\vec{r}_v = (0, 1, 2c(v - b)/b^2)$, and $\vec{r}_u \times \vec{r}_v = (0, -2c(v - b)/b^2, 1)$. Then the surface integral is

$$\begin{aligned}
&\iint_{D^\circ} \overrightarrow{F}(\vec{r}(u, v)) \cdot (\vec{r}_u \times \vec{r}_v) du\, dv \\
&\quad = \iint_{D^\circ} (c(v - b)^2/b^2, 0, v) \cdot (0, -2c(v - b)/b^2, 1) du\, dv \\
&\quad = \int_{u=0}^{u=a} \int_{v=0}^{v=b} v \, dv \, du.
\end{aligned}$$

   (c) $\frac{1}{2}b^2 a$.