Math 324A **Advanced Multivariable Calculus I** **Spring 1997**

TEST #3 SOLUTIONS

Outcome: Median for 324A & B: 84.5 (3.6). Breakdown: 90–100 (4.0): 36%; 75–89 (3.0–4.0): 35%; 60–74 (2.0–3.0): 18%; 45–59 (1.0–2.0): 5.5%; 0–44 (0.0): 5.5%.

Answers:

#1 (average score 17/20)

There are no shortcuts for this one: $\overrightarrow{F}$ is not conservative (by the curl test), and the tangential component of $\overrightarrow{F}$ is not constant, so your only choice is to parametrize the line segment and compute the line integral directly. One parametrization is given by $\vec{r}(t) = (2-2t, 0, 3t)$, with $0 \le t \le 1$, so

$$\int_C \overrightarrow{F} \cdot d\vec{r} = \int_0^1 \overrightarrow{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt = \int_0^1 (2-2t, 9t^2, 0) \cdot (-2, 0, 3)dt = \int_0^1 (-4+4t)dt = -2.$$

#2 (average score 18/20)

The easiest way to do this is to notice that curl $\overrightarrow{F} = 0$, and $\overrightarrow{F}$ and its partial derivatives are defined everywhere in the plane, so $\overrightarrow{F}$ is conservative. A potential is $g(x,y) = xy^2$, so the integral is equal to $g(2,4) - g(0,0) = 2(2)(4^2) = 32$.

This one can also be done directly. The parametrization of the curve $\vec{r}(t) = (t, t^2)$ is already given, so you just have to notice that the correct parameter interval is $0 \le t \le 2$, and so the integral is

$$\int_C \overrightarrow{F} \cdot d\vec{r} = \int_0^2 (t^4, 2t^3) \cdot (1, 2t)dt = \int_0^2 (t^4 + 4t^4)dt = \int_0^2 5t^4\,dt = t^5\Big|_0^2 = 32.$$

#3 (average score 15/20)

(a) Since $\overrightarrow{G}$ is given to you as a gradient, it is certainly conservative, and its potential is the given function $g(x,y) = e^{2x+y}$. Therefore $\int_C \overrightarrow{G} \cdot d\vec{r} = g(\pi, 0) - g(0,0) = e^{2\pi} - 1$.

(b) You're told that curl $\overrightarrow{F} = 0$. Because the denominator $1 + x^2y^2$ is always strictly positive, $\overrightarrow{F}$ and its partial derivatives are defined everywhere in the plane, so the curl test is sufficient to conclude that $\overrightarrow{F}$ is conservative. Finding a potential turns out to be pretty difficult. (You'd have to integrate $y/(1+x^2y^2)$ with respect to x: how many of you remember how to do that? For the record, the potential is $\arctan(xy)$.) Also, computing the integral by parametrizing the curve is out of the question, because you'd have to integrate $\frac{\sin t + t\cos t}{1+t^2\sin^2 t}$. But since $\overrightarrow{F}$ is conservative, you can replace the path C by a simpler one: the obvious path is along the x-axis, parametrized by $\vec{r}(t) = (t, 0)$ from $t = 0$ to $t = \pi$. Using this, you get

$$\int_C \overrightarrow{F} \cdot d\vec{r} = \int_0^\pi \left(\frac{0}{1+0}, \frac{t}{1+0}\right) \cdot (1,0)dt = \int_0^\pi 0\,dt = 0.$$

#4 (average score 14/20)

(a) The tangential component is the scalar projection of $\overrightarrow{F}$ onto $\vec{r}\,'(t)$, or

$$F_{tan} = \overrightarrow{F}(\vec{r}(t)) \cdot \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = (e^{4t^2}, e^0, e^{4t^2}) \cdot \frac{(2,0,1)}{\sqrt{5}} = \frac{3e^{4t^2}}{\sqrt{5}}.$$

(b) The upper and lower bounds are $k = (Min\ F_{tan})\,\text{Length}(L)$, $K = (Max\ F_{tan})\,\text{Length}(L)$. The minimum and maximum of F_{tan} are easily determined from part (a): since e^{4t^2} is an increasing function of t, the minimum is obtained at $t = 0$ and the maximum at $t = 1$. (If you didn't know it was an increasing function, you could have taken the derivative of e^{4t^2} and found that it is never zero for $0 < t < 1$, so the min and max have to occur at the endpoints by elementary calculus.) The length of L is $\sqrt{5}$, so the constants are

$$k = \left(\frac{3e^0}{\sqrt{5}}\right)\sqrt{5} = 3; \qquad K = \left(\frac{3e^4}{\sqrt{5}}\right)\sqrt{5} = 3e^4.$$

#5 (average score 16/20)

(a) Because its curl is nonzero, $\overrightarrow{G}$ cannot be conservative: the answer is NO.

(b) The curl is zero, and the component functions of $\overrightarrow{H}$ and their partial derivatives are defined everywhere in space, so $\overrightarrow{H}$ is conservative: the answer is YES. A potential is $xe^z + yz$.

(c) The curl is zero, so $\overrightarrow{K}$ might be conservative. However, the first two component functions (and their partial derivatives) are undefined anywhere that $x = y$, which includes infinitely many points; thus the curl test is inconclusive, and the correct answer is ???. [Some of you actually computed that a potential for $\overrightarrow{K}$ is $\log|x - y| + z^2$, so in fact it is conservative. If you did this correctly, I reluctantly gave you full credit even though you didn't, strictly speaking, answer the question that was asked.]