Math 324B **Advanced Multivariable Calculus I** **Spring 1997**

TEST #2 SOLUTIONS

Outcome: Median for 324A & B: 81 (3.4). Breakdown: 90–100 (4.0): 36%; 75–89 (3.0–4.0): 26%; 60–74 (2.0–3.0): 19%; 45–59 (1.0–2.0): 14%; 0–44 (0.0): 4%.

Answers:

#1 (a) $x'(t) = 2$; $y'(t) = x(t)$; $z'(t) = z(t)$. (b) The general solution to the system above is $x(t) = 2t + a$; $y(t) = t^2 + at + b$; $z(t) = ce^t$ (you have to plug in $x(t) = 2t + a$ to solve for y). Setting $(x, y, z) = (2, 1, 1)$ and $t = 2$, we can solve for $(a, b, c) = (-2, 1, e^{-2})$ to obtain $x(t) = 2t - 2$, $y(t) = t^2 - 2t + 1$, $z(t) = e^{t-2}$.

#2 (a) $\nabla \times \overrightarrow{G} = (0, -2xz^3, 2xy - x)$, and $\operatorname{div}(\nabla \times \overrightarrow{G}) = 0$ (a scalar). (b) $\operatorname{grad} f = (e^y - 2xz, xe^y, -x^2)$, and $\nabla \cdot (\operatorname{grad} f) = -2z + xe^y$. (c) BAD QUESTION: $\nabla \cdot \overrightarrow{G}$ is a scalar field (the divergence of $\overrightarrow{G}$), and you can't take the curl of a scalar field. (d) $\operatorname{div} \overrightarrow{G} = y + x^2 + 3x^2z^2$, and $\operatorname{grad}(\operatorname{div} \overrightarrow{G}) = (2x + 6xz^2, 1, 6x^2z)$.

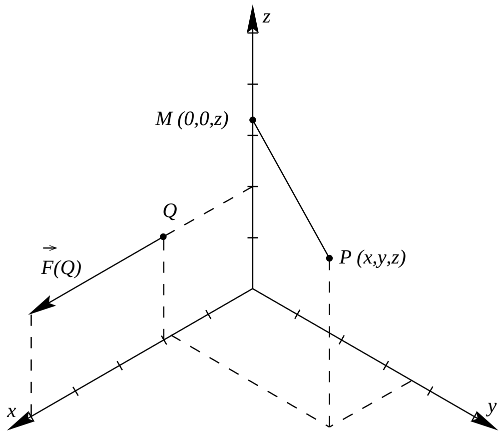
#3 (a) You have to notice that the vector field always points exactly away from the origin, so the flow lines will be rays pointing away from O. Since $\overrightarrow{F}$ has constant length equal to $\frac{1}{2}$, the rays will have constant-speed parametrizations, with speed $\frac{1}{2}$ and velocity equal to $\overrightarrow{F}$. We first need to compute

$$\overrightarrow{F}(R) = \frac{1}{2}\frac{\overrightarrow{OR}}{|\overrightarrow{OR}|} = \frac{1}{2}\frac{(3,0,4)}{\sqrt{3^2+4^2}} = \frac{1}{2}\frac{(3,0,4)}{5} = (0.3, 0, 0.4).$$

Then, since the motion has constant velocity, the position of the point after 1 second is just $\overrightarrow{OQ} = \overrightarrow{OR} + \overrightarrow{F}(R) = (3, 0, 4) + (0.3, 0, 0.4) = (3.3, 0, 4.4)$. (b) We already observed that the flow line is a constant-speed line. Since the initial point is $(3, 0, 4)$ and we just computed the velocity to be $(0.3, 0, 0.4)$, the answer is $x(t) = 3 + (0.3)t$, $y(t) = 0$, $z(t) = 4 + (0.4)t$.

#4 (a) The Key Fact about flow lines is: *velocity of the flow line equals field vector at each point.* Thus $\overrightarrow{F}(0, 0, 1) = \overrightarrow{F}(\vec{r}(0)) = \vec{r}\,'(0) = (4, 2t-1, 3t^2)|_{t=0} = (4, -1, 0)$. (b) The gradient of any function is orthogonal to its level objects. Since $\overrightarrow{F} = \operatorname{grad} f$, this means $\overrightarrow{F}(0, 0, 1)$ is orthogonal to the level surface of f through $P(0, 0, 1)$, which is the same as being orthogonal to the tangent plane. Therefore α is a plane whose normal vector is $\overrightarrow{F}(0, 0, 1) = (4, -1, 0)$ (from part (a)), so it has an equation of the form $4x - y = D$. The plane has to pass through $(0, 0, 1)$, so plugging in $(x, y, z) = (0, 0, 1)$ we find $D = 0$. The equation of α, therefore, is $4x - y = 0$.

#5 (a) See below.



(b) $\overrightarrow{F}(P)$ is a vector with the same direction as $\overrightarrow{MP} = (x, y, 0)$, but with length 3, or $\overrightarrow{F}(P) = 3\overrightarrow{MP}/|\overrightarrow{MP}|$. Thus $F_1(x, y, z) = 3x/\sqrt{x^2 + y^2}$, $F_2(x, y, z) = 3y/\sqrt{x^2 + y^2}$, $F_3(x, y, z) = 0$.