**Math 324A** **Advanced Multivariable Calculus I** **Spring 1997**

# Test #2 Solutions

**Outcome:** Median for 324A & B: 81 (3.4). Breakdown: 90–100 (4.0): 36%; 75–89 (3.0–4.0): 26%; 60–74 (2.0–3.0): 19%; 45–59 (1.0–2.0): 14%; 0–44 (0.0): 4%.

**Answers:**

**#1** (a) $x'(t) = x(t)$; $y'(t) = 2$; $z'(t) = y(t)$. (b) The general solution to the system above is $x(t) = ae^t$; $y(t) = 2t + b$; $z(t) = t^2 + bt + c$ (you have to plug in $y(t) = 2t + b$ to solve for $z$). Setting $(x, y, z) = (1, 2, 1)$ and $t = 2$, we can solve for $(a, b, c) = (e^{-2}, -2, 1)$ to obtain $x(t) = e^{t-2}$, $y(t) = 2t - 2$, $z(t) = t^2 - 2t + 1$.

**#2** (a) BAD QUESTION: $\nabla f$ is a vector field (the gradient of $f$), and you can't take the gradient of a vector field. (b) $\operatorname{grad} f = (e^z - y^2, -2xy, xe^z)$, and $\nabla \cdot (\operatorname{grad} f) = -2x + xe^z$. (c) $\operatorname{div} \overrightarrow{G} = 2xy^2 + z + y^3$, and $\operatorname{grad}(\operatorname{div} \overrightarrow{G}) = (2y^2, 4xy + 3y^2, 1)$. (d) $\nabla \times \overrightarrow{G} = (3y^2z - y, 0, -2x^2y)$, and $\operatorname{div}(\nabla \times \overrightarrow{G}) = 0$ (a scalar).

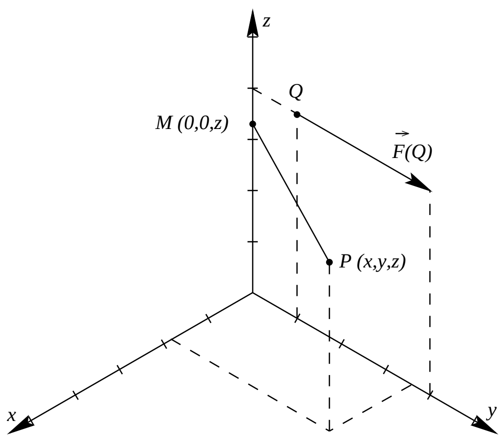
**#3** (a) You have to notice that the vector field always points exactly away from the origin, so the flow lines will be rays pointing away from $O$. Since $\overrightarrow{F}$ has constant length equal to $\frac{1}{2}$, the rays will have constant-speed parametrizations, with speed $\frac{1}{2}$ and velocity equal to $\overrightarrow{F}$. We first need to compute

$$\overrightarrow{F}(R) = \frac{1}{2}\frac{\overrightarrow{OR}}{|\overrightarrow{OR}|} = \frac{1}{2}\frac{(4,3,0)}{\sqrt{4^2+3^2}} = \frac{1}{2}\frac{(4,3,0)}{5} = (0.4, 0.3, 0).$$

Then, since the motion has constant velocity, the position of the point after 1 second is just $\overrightarrow{OQ} = \overrightarrow{OR} + \overrightarrow{F}(R) = (4, 3, 0) + (0.4, 0.3, 0) = (4.4, 3.3, 0)$. (b) We already observed that the flow line is a constant-speed line. Since the initial point is $(4, 3, 0)$ and we just computed the velocity to be $(0.4, 0.3, 0)$, the answer is $x(t) = 4 + (0.4)t$, $y(t) = 3 + (0.3)t$, $z(t) = 0$.

**#4** (a) The Key Fact about flow lines is: *velocity of the flow line equals field vector at each point.* Thus $\overrightarrow{F}(0, 0, 1) = \overrightarrow{F}(\vec{r}(0)) = \vec{r}\,'(0) = (1, 3t^2 - 4, 2t)|_{t=0} = (1, -4, 0)$. (b) The gradient of any function is orthogonal to its level objects. Since $\overrightarrow{F} = \operatorname{grad} f$, this means $\overrightarrow{F}(0, 0, 1)$ is orthogonal to the level surface of $f$ through $P(0, 0, 1)$, which is the same as being orthogonal to the tangent plane. Therefore $\alpha$ is a plane whose normal vector is $\overrightarrow{F}(0, 0, 1) = (1, -4, 0)$ (from part (a)), so it has an equation of the form $x - 4y = D$. The plane has to pass through $(0, 0, 1)$, so plugging in $(x, y, z) = (0, 0, 1)$ we find $D = 0$. The equation of $\alpha$, therefore, is $x - 4y = 0$.

**#5** (a) See below.




(b) $\overrightarrow{F}(P)$ is a vector with the same direction as $\overrightarrow{MP} = (x, y, 0)$, but with length 3, or $\overrightarrow{F}(P) = 3\overrightarrow{MP}/|\overrightarrow{MP}|$. Thus $F_1(x, y, z) = 3x/\sqrt{x^2 + y^2}$, $F_2(x, y, z) = 3y/\sqrt{x^2 + y^2}$, $F_3(x, y, z) = 0$.