Math 324B **Advanced Multivariable Calculus I** **Spring 1997**

FINAL EXAM SOLUTIONS

Outcome: 144–160 (4.0): 17%; 120–143 (3.0–3.9): 31%; 96–119 (2.0–2.9): 27%; 72–95 (1.0–1.9): 12%; under 72 (0.0): 13%. Median exam score 115; median course grade 3.2.

Answers:

1. (Average: 12/20) We must first find $\overrightarrow{F} = \operatorname{grad} g = (2x, -4y)$.

(a) The system of ODEs is $x'(t) = 2x(t)$; $y'(t) = -4y(t)$.

(b) The general solution is $(x(t), y(t)) = (Ae^{2t}, Be^{-4t})$ for some constants A, B. The specific solution with $(x, y) = (2, 1)$ when $t = 0$ is $x(t) = 2e^{2t}$ and $y(t) = e^{-4t}$.

(c) There is no such time. Since F is a gradient field, it and its flow lines are everywhere perpendicular to the level curves of g.

2. (Average: 15/20)

(a) $\vec{r}(u, v) = (u, v, uv - u + v)$.

(b) The point $P(1, 2, 3)$ corresponds to $(u, v) = (1, 2)$. A normal to S at P is given by $\vec{r}_u \times \vec{r}_v$ evaluated at P. We compute: $\vec{r}_u = (1, 0, v - 1)$, $\vec{r}_v = (0, 1, u + 1)$, and $\vec{r}_u \times \vec{r}_v = (1 - v, -u - 1, 1)$. At P, this is $\vec{n} = (-1, -2, 1)$

(c) We simply plug in $v = 2$ (the value of v at P) to our parameter representation for S: $(x, y, z) = (u, 2, 2u - u + 2) = (u, 2, u + 2)$.

3. (Average: 18/20) One possible setup for the integral is

$$\int_{x=0}^{x=3} \int_{y=0}^{y=1} \int_{z=0}^{z=y^2-2y+2} y^2 - 2y \; dz \; dy \; dx.$$

It can also be set up as $dz\,dx\,dy$ or $dx\,dz\,dy$; in both cases the limits are the same.

4. (Average: 17/20)

(a) A parameter representation for M is $\vec{r} = (1, 4, 0) + t(1, -4, 0)$. Thus

$$\int_M \overrightarrow{F} \cdot d\vec{r} = \int_0^1 \overrightarrow{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) \; dt = \int_0^1 (0, 4 - 4t, \sin(1 + t)) \cdot (1, -4, 0) \; dt$$
$$= \int_0^1 -4(4 - 4t) \; dt = -8.$$

(b) By Stokes's Theorem, we have $\int_E \overrightarrow{F} \cdot d\vec{r} = \iint_S \operatorname{curl} \overrightarrow{F} \cdot d\overrightarrow{A}$, where $S = ABCDA$. Since $\operatorname{curl} \overrightarrow{F}$ is constant, we can use the normal component shortcut to get

$$\iint_S \operatorname{curl} \overrightarrow{F} \cdot d\overrightarrow{A} = \operatorname{curl} \overrightarrow{F} \cdot \frac{\vec{n}}{|\vec{n}|} A(S) = \operatorname{curl} \overrightarrow{F} \cdot \frac{\overrightarrow{AB} \times \overrightarrow{BC}}{|\overrightarrow{AB} \times \overrightarrow{BC}|} |\overrightarrow{AB} \times \overrightarrow{BC}|$$
$$= \operatorname{curl} \overrightarrow{F} \cdot (\overrightarrow{AB} \times \overrightarrow{BC}) = 12.$$

Note that the vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are chosen so that their cross product produces a normal consistent with the orientation of E.

5. (Average: 16/20) Because the paddlewheel is small, curl $\overrightarrow{F}$ is almost constant along it, so Stokes's theorem tells us that the circulation is approximately $F_n A(S) = (\text{curl } \overrightarrow{F} \cdot \vec{n}/|\vec{n}|)A(S)$. To determine the sign of the circulation, we just have to compute curl $\overrightarrow{F} \cdot \vec{n}$.

(a) Compute curl $\overrightarrow{F}(P) \cdot \vec{n} = (-1, 14, -4) \cdot (1, 1, 3) = 1 > 0$. This implies the circulation is positive, so the paddle wheel will spin *counterclockwise*.

(b) At such a point Q, we must have curl $\overrightarrow{F}(Q) = \vec{0}$. This means that $(x, y, z) = (2, \pm 3, -3)$.

6. (Average: 16/20)

(a) Since curl $\overrightarrow{F} = 2 \neq 0$, $\overrightarrow{F}$ is **not** conservative.

(b) Since curl $\overrightarrow{G} = (0, 0, 0)$ and the derivatives of $\overrightarrow{G}$ are defined everywhere except $(2, 0, -2)$ and $(0, 3, 0)$, the curl test implies that $\overrightarrow{G}$ is conservative.

(c) The vector field $\overrightarrow{K}$ is conservative by the curl test. One potential is $g(x, y, z) = e^x - x^2 y + 3yz$. Thus the integral along a path C from $(0, 0, 0)$ to $(1, 2, 3)$ is $\int_C \overrightarrow{K} \cdot d\vec{r} = g(1, 2, 3) - g(0, 0, 0) = e + 15$.

7. (Average: 12/20) A normal for S is given by $\vec{n} = (x, y, z)$, so the normal component of $\overrightarrow{F}$ is $F_n = \overrightarrow{F} \cdot \frac{\vec{n}}{|\vec{n}|} = (z^2 + 1)b$ (notice that $\sqrt{x^2 + y^2 + z^2} = b$ for points on S). Thus, on the hemisphere S, we have the bounds $b \leq F_n \leq (b^2 + 1)b$. Since the surface area of S is $2\pi b^2$ (half the area of the sphere!), we get $2\pi b^3 \leq \text{flux} \leq 2\pi b^3(b^2 + 1)$.

8. (Average: 13/20)

(a) The simplest way to do this is to parametrize using cylindrical coordinates: $\vec{r}(u, v) = (b\cos u, b\sin u, v)$. We compute $\vec{r}_u = (-b\sin u, b\cos u, 0)$, $\vec{r}_v = (0, 0, 1)$, and $\vec{r}_u \times \vec{r}_v = (b\cos u, b\sin u, 0)$. Then

$$\iint_S \overrightarrow{F} \cdot d\overrightarrow{A} = \int_{v=0}^{v=a} \int_{u=0}^{u=2\pi} (vb\cos u, vb\sin u, e^{b^2}) \cdot (b\cos u, b\sin u, 0)\; du dv = \pi a^2 b^2.$$

You could in principle use the divergence theorem, but since S is not closed you'd have to subtract off surface integrals over T and B, and end up doing more work.

(b) Let K be the solid bounded by E. Then by the divergence theorem,

$$\iint_E \overrightarrow{G} \cdot d\overrightarrow{A} = \iiint_K \text{div } \overrightarrow{G}\; dV.$$

Since div $\overrightarrow{G} = 3$, the integral on the right is just 3 times the volume of K (which is half the volume of the sphere), or $3\pi ab^2$. This is the flux.