Math 324A **Advanced Multivariable Calculus I** **Spring 1997**

FINAL EXAM SOLUTIONS

Outcome: 144–160 (4.0): 17%; 120–143 (3.0–3.9): 31%; 96–119 (2.0–2.9): 27%; 72–95 (1.0–1.9): 12%; under 72 (0.0): 13%. Median exam score 115; median course grade 3.2.

Answers:

1. (Average: 15/20)

(a) Since curl $\overrightarrow{G} = 0$, it might be conservative. But the derivatives of $\overrightarrow{G}$ are not defined at the points $(2,0)$ and $(0,3)$, so the test is inconclusive.

(b) Since curl $\overrightarrow{F} \neq 0$, $\overrightarrow{F}$ is **not** conservative.

(c) The vector field $\overrightarrow{K}$ is conservative by the curl test. One potential is $g(x,y,z) = \sin x + y^2 - z^2 y$. Thus the integral along a path C from $(0,0,0)$ to $(\pi, 2\pi^2, 3\pi)$ is $\int_C \overrightarrow{K} \cdot d\vec{r} = g(\pi, 2\pi^2, 3\pi) - g(0,0,0) = -14\pi^4$.

2. (Average: 14/20)

(a) A parameter representation for M is $\vec{r} = (2,0,0) + t(1,0,3)$. Thus

$$\int_M \overrightarrow{F} \cdot d\vec{r} = \int_0^1 \overrightarrow{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt = \int_0^1 (-3t, (3t-1)e^{3t}, 0) \cdot (1,0,3)\, dt$$
$$= \int_0^1 (-3t)\, dt = -\frac{3}{2}.$$

(b) By Stokes's Theorem, we have $\oint_E \overrightarrow{F} \cdot d\vec{r} = \iint_S \text{curl}\, \overrightarrow{F} \cdot d\overrightarrow{A}$, where $S = ABCDA$. Since curl $\overrightarrow{F}$ is constant, we can use the normal component shortcut to get

$$\iint_S \text{curl}\, \overrightarrow{F} \cdot d\overrightarrow{A} = \text{curl}\, \overrightarrow{F} \cdot \frac{\vec{n}}{|\vec{n}|} A(S) = \text{curl}\, \overrightarrow{F} \cdot \frac{\overrightarrow{AB} \times \overrightarrow{BC}}{|\overrightarrow{AB} \times \overrightarrow{BC}|} |\overrightarrow{AB} \times \overrightarrow{BC}|$$
$$= \text{curl}\, \overrightarrow{F} \cdot (\overrightarrow{AB} \times \overrightarrow{BC}) = 3.$$

Note that the vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are chosen so that their cross product produces a normal consistent with the orientation of E.

3. (Average: 11/20) A normal for S is given by $\vec{n} = (x,y,z)$.

(a) The normal component of $\overrightarrow{F}$ is $F_n = \overrightarrow{F} \cdot \vec{n}/|\vec{n}| = (z + 2z^2)/2$ (notice that $\sqrt{x^2+y^2+z^2} = 2$ for points on S). Thus, on the hemisphere S, we have the bounds $0 \leq F_n \leq 5$. Since the surface area of S is 8π (half the area of the sphere!), we get $0 \leq \text{flux} \leq 40\pi$.

(b) By the divergence theorem, $\iint_T \overrightarrow{F} \cdot d\vec{A} = \iiint_K \text{div}\, \overrightarrow{F}\, dV$, where K is the solid upper hemisphere. Since div $\overrightarrow{F} = 2$ is constant, this is just $2\,Vol(K) = 32\pi/3$.

4. (Average: 12/20) We must first find $\overrightarrow{F} = \operatorname{grad} g = (2x, -4y)$.

(a) The system of ODEs is $x'(t) = 2x(t)$; $y'(t) = -4y(t)$.

(b) The general solution is $(x(t), y(t)) = (Ae^{2t}, Be^{-4t})$ for some constants A, B. The specific solution with $(x, y) = (2, 1)$ when $t = 0$ is $x(t) = 2e^{2t}$ and $y(t) = e^{-4t}$.

(c) There is no such time. Since F is a gradient field, it and its flow lines are everywhere perpendicular to the level curves of g.

5. (Average: 18/20) One possible setup for the integral is

$$\int_{y=0}^{y=6} \int_{x=0}^{x=2} \int_{z=0}^{z=3-\frac{1}{4}x^2} x^2 + 4x \; dz \; dx \; dy.$$

It can also be set up as $dz\,dy\,dx$ or $dy\,dz\,dx$; in both cases the limits are the same.

6. (Average: 8/20) Because the circle C is small, curl $\overrightarrow{F}$ is almost constant along it, so Stokes's theorem tells us that the circulation is approximately $F_n A(S) = (\operatorname{curl} \overrightarrow{F} \cdot \vec{n}/|\vec{n}|)A(S)$. To determine the sign of the circulation, we just have to compute curl $\overrightarrow{F} \cdot \vec{n}$.

(a) Compute curl $\overrightarrow{F}(P) \cdot \vec{n} = (0, 12, -8) \cdot (0, 0, 1) = -8 < 0$.

(b) The maximum circulation occurs when $\overrightarrow{F} \cdot \vec{n}/|\vec{n}|$ is maximum, which occurs when $\overrightarrow{F}$ is parallel to $\vec{n}$. Thus α is the plane through the origin whose normal is $(0, 12, -8)$, or $12y - 8z = 0$.

7. (Average: 12/20)

(a) Using cylindrical coordinates, $\vec{r}(u, v) = (b \cos u, b \sin u, v)$.

(b) Part (a) was supposed to be a hint that you should do this by parametrization. Compute $\vec{r}_u = (-b \sin u, b \cos u, 0)$, $\vec{r}_v = (0, 0, 1)$, and $\vec{r}_u \times \vec{r}_v = (b \cos u, b \sin u, 0)$. Then

$$\iint_S \overrightarrow{F} \cdot d\vec{A} = \int_{v=0}^{v=a} \int_{u=0}^{u=2\pi} (v^2 b \cos u, v^2 b \sin u, e^{b^2 \cos^2 u}) \cdot (b \cos u, b \sin u, 0) \, dudv,$$

which evaluates to $\frac{2}{3}\pi a^3 b^2$. You could in principle use the divergence theorem, but since S is not closed you'd have to subtract off surface integrals over T and B, and end up doing more work.

8. (Average: 15/20)

(a) $\vec{r}(u, v) = (u, v, e^u + v - uv^2)$.

(b) The point $P(1, 2, 3)$ corresponds to $(u, v) = (0, 1)$. A normal to S at P is given by $\vec{r}_u \times \vec{r}_v$ evaluated at P. Compute: $\vec{r}_u = (1, 0, e^u - v^2) = (1, 0, 0)$ at P, $\vec{r}_v = (0, 1, 1 - 2uv) = (0, 1, 1)$ at P, and $\vec{r}_u \times \vec{r}_v = (0, -1, 1)$.

(c) We simply plug in $v = 1$ (the value of v at P) to our parameter representation for S: $(x, y, z) = (u, 1, e^u + 1 - u)$.