Math 310 **Introduction to Mathematical Reasoning** **Spring 2006**

Handout #6: Solutions to Problem 23(i–ii), page 56

(i) Theorem: *For any nonzero real numbers x, y and any integer n, $x^n y^n = (xy)^n$.*

Proof: The case $n \geq 0$ is taken care of by Exercise 5.7(i). For the case in which $n < 0$, let $k = -n$, so $k > 0$.

Notice first that by the definition of negative exponents, we have $x^{-k} = (x^k)^{-1}$. It follows that

$$x^{-k}x^k = 1, \tag{1}$$

and similarly that

$$y^{-k}y^k = 1, \tag{2}$$

$$(xy)^{-k}(xy)^k = 1. \tag{3}$$

By the result of Exercise 5.7(i), we know that

$$(xy)^k = x^k y^k.$$

Multiplying both sides of this equation by $(xy)^{-k}x^{-k}y^{-k}$ and using (1)–(3) to simplify, we obtain

$$x^{-k}y^{-k} = (xy)^{-k}.$$

Substituting $k = -n$, we obtain the desired equation. □

(ii) Theorem: *For any nonzero real x and any integers m and n, $x^{m+n} = x^m x^n$.*

Proof: There are four cases, depending on the signs of m and n.

The case in which $m \geq 0$ and $n \geq 0$ is taken care of by Exercise 5.7(ii).

For the second case, assume that $m \geq 0$ and $n < 0$. Let $k = -n$, so what we have to prove is

$$x^{m-k} = x^m x^{-k}. \tag{4}$$

We consider two possibilities: either $m \geq k$ or $m < k$. If $m \geq k$, Exercise 5.7(ii) yields

$$x^{m-k}x^k = x^m.$$

Multiplying both sides by x^{-k} and simplifying (using equation (1) from the preceding proof), we obtain (4). On the other hand, if $m < k$, we apply Exercise 5.7(ii) again to obtain

$$x^k = x^m x^{k-m}.$$

Multiplying both sides by $x^{-k}x^{m-k}$ and simplifying once again yields (4).

The third case, in which $n \geq 0$ and $m < 0$, is handled in exactly the same way, with the roles of m and n reversed.

Finally, for the last case, assume that $m < 0$ and $n < 0$. Let $k = -m$ and $l = -n$, so what we have to prove is

$$x^{-k-l} = x^{-k}x^{-l}. \tag{5}$$

We use Exercise 5.7(ii) one more time to conclude that

$$x^k x^l = x^{k+l}.$$

Multiplying both sides by $x^{-k-l}x^{-k}x^{-l}$ and simplifying, we obtain (5) as desired. □

You may turn in a paragraph-style proof of part (iii) for extra credit if you wish.