Math 310 **Introduction to Mathematical Reasoning** **Spring 2006**

Handout #5: Common Properties of Numbers (CORRECTED)

These are some of the elementary properties of numbers that can be proved from the axioms on Handout #1. These can be proved in the order listed—that is, each statement can be proved using only the axioms and statements that appear before it on the list.

In all of the properties below, a, b, c, d represent arbitrary real numbers unless otherwise specified.

1. Properties of zero
   (a) $0 \times a = 0$.
   (b) If $ab = 0$, then $a = 0$ or $b = 0$.
2. Properties of signs
   (a) $-0 = 0$.
   (b) $-(-a) = a$.
   (c) $-a = (-1)a$.
   (d) $(-a)b = -(ab) = a(-b)$.
   (e) $(-a)(-b) = ab$.
3. Extensions of the distributive property
   (a) $-(a + b) = (-a) + (-b) = -a - b$.
   (b) $-(a - b) = b - a$.
   (c) $-(-a - b) = a + b$.
   (d) $a + a = 2a$.
   (e) $a(b - c) = ab - ac = (b - c)a$.
   (f) $(a + b)(c + d) = ac + ad + bc + bd$.
   (g) $(a + b)(c - d) = ac - ad + bc - bd = (c - d)(a + b)$.
   (h) $(a - b)(c - d) = ac - ad - bc + bd$.
4. Properties of inverses
   (a) $1^{-1} = 1$.
   (b) $(a^{-1})^{-1} = a$ if a is nonzero.
   (c) $(-a)^{-1} = -(a^{-1})$ if a is nonzero.
   (d) $(ab)^{-1} = a^{-1}b^{-1}$ if a and b are nonzero.
   (e) $(a/b)^{-1} = b/a$ if a and b are nonzero.
5. Properties of quotients
   (a) $a/1 = a$.
   (b) $(a/b)(c/d) = (ac)/(bd)$ if b and d are nonzero.
   (c) $(a/b)/(c/d) = (ad)/(bc)$ if b, c, and d are nonzero.
   (d) $(ac)/(bc) = a/b$ if b and c are nonzero.
   (e) $(-a)/b = -(a/b) = a/(-b)$ if b is nonzero.
   (f) $(-a)/(-b) = a/b$ if b is nonzero.
   (g) $a/b + c/d = (ad + bc)/(bd)$ if b and d are nonzero.
   (h) $a/b - c/d = (ad - bc)/(bd)$ if b and d are nonzero.

6. PROPERTIES OF INEQUALITIES

   (a) $0 < 1$.
   (b) If $a < b$, then $-a > -b$.
   (c) If $a < b$ and a and b are both positive, then $a^{-1} > b^{-1}$.
   (d) If $a \leq b$ and $b \leq c$, then $a \leq c$.
   (e) If $a \leq b$ and $b < c$, then $a < c$.
   (f) If $a < b$ and $b \leq c$, then $a < c$.
   (g) If $a < b$ and $c < d$, then $a + c < b + d$.
   (h) If $a \leq b$ and $c < d$, then $a + c < b + d$.
   (i) If $a \leq b$ and $c \leq d$, then $a + c \leq b + d$.
   (j) If $a \leq b$ and $c > 0$, then $ac \leq bc$.
   (k) If $a \leq b$ and $c < 0$, then $ac \geq bc$.
   (l) If $a < b$ and $c < d$, and a, b, c, d are nonnegative, then $ac < bd$.
   (m) If $a < b$ and $c \leq d$, and a, b, c, d are nonnegative, then $ac < bd$.
   (n) If $a \leq b$ and $c \leq d$, and a, b, c, d are nonnegative, then $ac \leq bd$.
   (o) $ab > 0$ if and only if a and b are both positive or both negative.
   (p) $ab < 0$ if and only if one is positive and the other is negative.
   (q) If $a \leq b$ and $b \leq a$, then $a = b$.
   (r) There does not exist a smallest positive real number.

7. PROPERTIES OF SQUARES

   (a) If $a \neq 0$, then $a^2 > 0$.
   (b) For any a, $a^2 \geq 0$.
   (c) If a and b are positive, then $a < b \Rightarrow a^2 < b^2$.
   (d) If a and b are negative, then $a < b \Rightarrow a^2 > b^2$.

8. ORDER PROPERTIES OF INTEGERS

   (a) If n is a positive integer, then $n \geq 1$.
   (b) If m and n are integers such that $m > n$, then $m \geq n + 1$.
   (c) There does not exist a largest integer.

9. LAWS OF EXPONENTS
   In statements (a)–(d) *below, m and n are assumed to be nonnegative integers.*

   (a) $a^n b^n = (ab)^n$.
   (b) $a^{m+n} = a^m a^n$.
   (c) $(a^m)^n = a^{mn}$.
   (d) $a^n / b^n = (a/b)^n$ if b is nonzero.
   (e) Identities 9(a)–9(d) hold for arbitrary integers m and n, provided a and b are nonzero.

10. DIVISIBILITY PROPERTIES OF INTEGERS
   In each of the following statements, m, n, and p are assumed to be integers.

   (a) If $m|n$ and $n|p$, then $m|p$.
   (b) If $m|n$ and $m|p$, then $m|(n + p)$.
   (c) If $m|n$ or $m|p$, then $m|np$.
   (d) If n is even, so is n^2.
   (e) If n^2 is odd, so is n.
   (f) If m and n are even, so is $m + n$.
   (g) If m is even, then so is mn for any integer n.