Math 310 **Introduction to Mathematical Reasoning** **Spring 2006**

Handout #3: Undefined Terms, Definitions, and Axioms

UNDEFINED TERMS

To avoid circularity, we cannot give every term a rigorous mathematical definition; we have to accept some things as undefined terms. For this course, we will take the following fundamental notions as undefined terms. You already know what these terms mean; but the only facts about them that can be used in proofs are the ones expressed in the axioms listed below (and any theorems that can be proved from the axioms).

- ***Integer:*** Intuitively, an integer is a whole number (positive, negative, or zero).
- ***Real number:*** Intuitively, a real number represents a point on the number line, or a (signed) distance left or right from the origin, or any quantity that has a finite or infinite decimal representation. Real numbers include integers, positive and negative fractions, and irrational numbers like π, e, and $\sqrt{2}$.
- ***Zero:*** The number zero is denoted by $\boxed{0}$.
- ***One:*** The number one is denoted by $\boxed{1}$.
- ***Sum:*** The sum of two integers or real numbers a and b is denoted by $\boxed{a+b}$.
- ***Product:*** The product of two integers or real numbers a and b is denoted by $\boxed{ab}$ or $\boxed{a \cdot b}$ or $\boxed{a \times b}$.
- ***Less than:*** To say that a is less than b, denoted by $\boxed{a < b}$, means intuitively that a is to the left of b on the number line.

DEFINITIONS

In all the definitions below, a and b represent arbitrary real numbers.

- The set of all real numbers is denoted by $\boxed{\mathbb{R}}$, and the set of all integers is denoted by $\boxed{\mathbb{Z}}$.
- The $\boxed{\text{difference between } a \text{ and } b}$, denoted by $\boxed{a - b}$, is the real number defined by $a - b = a + (-b)$, where $-b$ is the additive inverse of b whose existence is guaranteed by Axiom 8.
- If $b \neq 0$, the $\boxed{\text{quotient of } a \text{ and } b}$, denoted by $\boxed{a/b}$, is the real number defined by $a/b = ab^{-1}$, where b^{-1} is the multiplicative inverse of b whose existence is guaranteed by Axiom 9.
- A real number is said to be $\boxed{\text{rational}}$ if it is equal to p/q for some integers p and q with $q \neq 0$. The set of all rational numbers is denoted by $\mathbb{Q}$.
- A real number is said to be $\boxed{\text{irrational}}$ if it is not rational.
- The phrase $\boxed{a \text{ is less than or equal to } b}$, denoted by $\boxed{a \leq b}$, means $a < b$ or $a = b$.
- The phrase $\boxed{a \text{ is greater than } b}$, denoted by $\boxed{a > b}$, means $b < a$.
- The phrase $\boxed{a \text{ is greater than or equal to } b}$, denoted by $\boxed{a \geq b}$, means $a > b$ or $a = b$.
- A real number a is said to be $\boxed{\text{positive}}$ if $a > 0$.
- A real number a is said to be $\boxed{\text{negative}}$ if $a < 0$.
- A real number a is said to be $\boxed{\text{nonnegative}}$ if $a \geq 0$.
- A real number a is said to be $\boxed{\text{nonpositive}}$ if $a \leq 0$.
- If S is a set of real numbers, a real number b is said to be an $\boxed{\text{upper bound}}$ for S if $b \geq a$ for every a in S. It is said to be a $\boxed{\text{least upper bound}}$ for S if every other upper bound b' for S satisfies $b' \geq b$.

AXIOMS

We assume that the following statements are true.

1. (Closure of $\mathbb{Z}$) If a and b are integers, then so are $a + b$ and ab.
2. (Closure of $\mathbb{R}$) If a and b are real numbers, then so are $a + b$ and ab.
3. (Commutativity) $a + b = b + a$ and $ab = ba$ for all real numbers a and b.
4. (Associativity) $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$ for all real numbers a, b, and c.
5. (Distributivity) $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$ for all real numbers a, b, and c.
6. (Zero) 0 is both a real number and an integer, which satisfies $a + 0 = a = 0 + a$ for every real number a.
7. (One) 1 is both a real number and an integer, which is not equal to zero, and satisfies $a \times 1 = a = 1 \times a$ for every real number a.
8. (Additive inverses) If a is any real number, there is a unique real number $-a$ such that $a+(-a) = 0$. If a is an integer, then so is $-a$.
9. (Multiplicative inverses) If a is any nonzero real number, there is a unique real number a^{-1} such that $a \times a^{-1} = 1$.
10. (Trichotomy law) If a and b are real numbers, then one and only one of the following three possibilities is true: $a < b$, $a = b$, or $a > b$.
11. (Addition law for inequalities) Suppose a, b, and c are real numbers. If $a < b$, then $a + c < b + c$.
12. (Multiplication law for inequalities) Suppose a, b, and c are real numbers, and $a < b$. If $c > 0$, then $ac < bc$. If $c < 0$, then $ac > bc$.
13. (Transitive law for inequalities) For real numbers a, b, and c, if $a < b$ and $b < c$, then $a < c$.
14. (The induction axiom) If S is a set of positive integers that contains 1, and that contains $n + 1$ whenever it contains n, then S contains all the positive integers.
15. (The completeness axiom) If S is any nonempty set of real numbers and S has an upper bound, then S has a least upper bound.