Math 310 B & C **Introduction to Mathematical Reasoning** **Autumn 2005**

Handout #1 (10/5/05)

The Natural Numbers, the Integers, and the Rationals

Throughout this course, we assume the existence and basic properties of the familiar number systems $\mathbb{N}$ (the natural numbers), $\mathbb{Z}$ (the integers), $\mathbb{Q}$ (the rational numbers), and $\mathbb{R}$ (the real numbers). On pages 379–381 in the textbook, there is a list of properties of $\mathbb{R}$, which you may take as axioms that can be used in any proofs in this course, unless otherwise specified.

The purpose of this handout is to supplement those properties of $\mathbb{R}$ with some basic properties of the other number systems. Of course, since all elements of $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ are also elements of $\mathbb{R}$, they satisfy all of the properties of real numbers given in the text, such as the commutative, associative, and distributive properties (A1, A2, M1, M2, and D1) and the properties of inequalities expressed in Theorem 28.1. However, it is useful to rephrase some of the properties when we're talking only about subsets of $\mathbb{R}$—for example, it is not true that every element of $\mathbb{N}$ has an additive inverse in $\mathbb{N}$, although it is still true that $m + n = m + p$ implies $n = p$ (see the cancellation law for addition below). More importantly, there are some other properties of these special number systems that do not obviously follow from properties of $\mathbb{R}$. This handout lists all the properties you should need. (Later, we will explore ways of proving these properties from a small number of axioms; but for now, you can simply take all of these properties as axioms.)

THE NATURAL NUMBERS

We assume the existence of a subset $\mathbb{N} \subseteq \mathbb{R}$, whose elements are called ***natural numbers***. They satisfy the following properties.

$\mathbb{N}$1. (Existence of additive and multiplicative identities) 0 and 1 are natural numbers.

$\mathbb{N}$2. (Closure) If m and n are natural numbers, then so are $m + n$ and $m \cdot n$.

$\mathbb{N}$3. (The cancellation law for addition) If m, n, and p are natural numbers such that $m+n = m+p$, then $n = p$.

$\mathbb{N}$4. (The cancellation law for multiplication) If m, n, and p are natural numbers such that $m \neq 0$ and $m \cdot n = m \cdot p$, then $n = p$.

$\mathbb{N}$5. (Positivity) If m is a nonzero natural number, then $m > 0$.

$\mathbb{N}$6. (Non-density) If m is a natural number, there is no natural number n such that $m < n < m + 1$.

$\mathbb{N}$7. (The well-ordering principle) Every nonempty subset of $\mathbb{N}$ contains a smallest element.

$\mathbb{N}$8. (The induction principle) Suppose S is a subset of $\mathbb{N}$ satisfying the following two properties:

(i) $0 \in S$.

(ii) Whenever n is an element of S, $n + 1$ is also in S.

Then S is all of $\mathbb{N}$.

In addition, because the natural numbers are also integers, they satisfy all the properties of the integers listed below.

THE INTEGERS

Definition 1: The subset $\mathbb{Z} \subseteq \mathbb{R}$ is defined by

$$\mathbb{Z} = \{n \in \mathbb{R} : n \in \mathbb{N} \text{ or } -n \in \mathbb{N}\}.$$

The elements of $\mathbb{Z}$ are called the ***integers***.

Definition 2: If n and d are integers with $d > 0$, we say that d ***divides*** n if there exists an integer q such that $n = d \cdot q$.

Definition 3: An integer p is said to be ***prime*** if $p > 1$ and the only positive integers that divide p are 1 and p.

Definition 4: Let m and n be integers. A positive integer d is called the ***greatest common divisor*** of m and n if d divides both m and n and d is greater than or equal to every other common divisor of m and n.

Definition 5: An integer n is said to be ***even*** if 2 divides n, and ***odd*** if not.

The integers satisfy the following properties.

$\mathbb{Z}1$. (Existence of additive and multiplicative identities) 0 and 1 are integers.

$\mathbb{Z}2$. (Existence of additive inverse) If $n \in \mathbb{Z}$, then $-n \in \mathbb{Z}$.

$\mathbb{Z}3$. (Closure) If m and n are integers, then so are $m + n$ and $m \cdot n$.

$\mathbb{Z}4$. (Non-density) If m is an integer, there is no integer n such that $m < n < m + 1$.

$\mathbb{Z}5$. (The division algorithm) If n and d are integers with $d > 0$, there exist unique integers q and r such that $n = d \cdot q + r$ and $0 \leq r < d$.

$\mathbb{Z}6$. (Even-odd representation) An integer n is even if and only if it can be written in the form $n = 2 \cdot k$ for some integer k, and odd if and only it can be written $n = 2 \cdot k + 1$ for some integer k.

$\mathbb{Z}7$. (Unique factorization) Any integer greater than 1 can be written in one and only one way (except for reordering) as a product of primes.

$\mathbb{Z}8$. (Prime factors of a product) If m and n are integers and p is a prime that divides $m \cdot n$, then p divides m or p divides n.

$\mathbb{Z}9$. (Existence of a greatest common divisor) Any two integers m and n, not both zero, have a unique greatest common divisor.

$\mathbb{Z}10$. (Archimedean property) If x is any real number, there exists an integer n such that $n > x$.

In addition, because the integers are also rational numbers, they satisfy all the properties of the rational numbers listed below.

THE RATIONAL NUMBERS

Definition 5: The subset $\mathbb{Q} \subseteq \mathbb{R}$ is defined by

$$\mathbb{Q} = \{x \in \mathbb{R} : x = p/q \text{ for some } p, q \in \mathbb{Z} \text{ with } q \neq 0\}.$$

The elements of $\mathbb{Q}$ are called the ***rational numbers***.

The rational numbers satisfy the following properties.

$\mathbb{Q}1$. (Existence of additive and multiplicative identities) 0 and 1 are rational numbers.

$\mathbb{Q}2$. (Existence of additive inverse) If $x \in \mathbb{Q}$, then $-x \in \mathbb{Q}$.

$\mathbb{Q}3$. (Existence of multiplicative inverse) If $x \in \mathbb{Q}$ and $x \neq 0$, then $x^{-1} \in \mathbb{Q}$.

$\mathbb{Q}4$. (Closure) If x and y are rational numbers, then so are $x + y$ and $x \cdot y$.

$\mathbb{Q}5$. (Density) If x and y are real numbers with $x < y$, there is a rational number z such that $x < z < y$.

$\mathbb{Q}6$. (Reduction to lowest terms) If x is any nonzero rational number, there exist unique integers p and q with $q > 0$ such that $x = p/q$ and the greatest common divisor of p and q is 1.

THE REAL NUMBERS

To the axioms on pages 379–381 of the textbook, we add the following:

$\mathbb{R}1$. (Completeness) Every nonempty subset of real numbers that is bounded above has a least upper bound.