Math 300 **Introduction to Mathematical Reasoning** **Autumn 2018**

Handout 8: Rational Numbers

Rational and irrational numbers are defined on the last page of our Axioms handout (Handout 4): A real number a is said to be ***rational*** if there are integers p and q with $q \neq 0$ such that $a = p/q$, and it is said to be ***irrational*** if it is not rational.

The following theorem expresses one of the most important facts about rational numbers. It is the main ingredient in the proof that $\sqrt{2}$ is irrational. (See Theorem 3.20 on pp. 124–125 of our textbook. The result of the following theorem is mentioned on page 124, but the book does not give a proof.)

Theorem 1. *Suppose x is a rational number. Then x can be expressed in the form $x = p_1/q_1$, where p_1 and q_1 are integers such that $q_1 > 0$ and p_1 and q_1 have no common factors greater than 1. Such an expression for x is said to be in **lowest terms**.*

Proof. Let x be an arbitrary rational number. Define a set S of positive integers as follows:

$$S = \{q \in \mathbb{Z}^+ : x = p/q \text{ for some integer } p\}.$$

(In other words, S is the set of all positive denominators that can be used to express x as a fraction.) Then S is a set of positive integers by definition; we wish to show that it is nonempty. The hypothesis that x is rational means there are some integers p, q with $q \neq 0$ such that $x = p/q$. If $q > 0$, then $q \in S$; while if $q < 0$, then $x = (-p)/(-q)$, so $-q \in S$. This shows that $S \neq \emptyset$.

The well-ordering axiom guarantees that S contains a smallest integer; call it q_1. The fact that $q_1 \in S$ means that $q_1 > 0$ and $x = p_1/q_1$ for some integer p_1. To complete the proof, we need to show that p_1 and q_1 have no common factors greater than 1.

Assume for the sake of contradiction that k is a common factor greater than 1. This means k divides both p_1 and q_1, so $p_1 = km$ and $q_1 = kn$ for some integers m and n. By algebra, this implies

$$x = \frac{p_1}{q_1} = \frac{km}{kn} = \frac{m}{n}.$$

On the other hand, the fact that $k > 1$ means that $n > 0$ (for otherwise $q_1 = kn$ would be negative or zero, which is a contradiction). Thus we have shown that n is a positive integer and $x = m/n$ for some integer m, which is to say that $n \in S$. However, the inequality $k > 1$ can be multiplied on both sides by the positive integer n, yielding $kn > n$, from which we conclude that

$$q_1 = kn > n,$$

contradicting the fact that q_1 is the smallest element of S. □