Math 300A **Introduction to Mathematical Reasoning** **Autumn 2018**

Handout 3: Mathematical Statements

The basic building blocks of mathematical theorems and proofs are ***mathematical statements***. These are assertions of facts about "mathematical objects"—things like numbers, points, lines, ordered pairs, vectors, sets, functions, triangles—the usual flora and fauna of mathematical discourse.

Most importantly, a mathematical statement must be *unambiguous*, and it must be *either true or false, but not both.* (For any particular statement, we might not *know* whether it is true or false, but it must in principle be one or the other.) Thus an English sentence such as "This statement is false" cannot be considered as a mathematical statement, because if it were true, we would have to conclude that it was false; and if it were false, we would have to conclude that it was true. Similarly, "The number 42 is interesting" cannot be considered as a mathematical statement unless we are prepared to give an unambiguous definition of the concept of an "interesting number."

Complicated mathematical statements are all built up in a systematic way from simpler ones. The basic building blocks are ***atomic statements*** (also called ***simple statements***), which are ones that cannot be expressed as combinations of smaller statements. They can be written using mathematical symbols, or English words, or some combination of the two. Some examples of simple statements are "$\pi > 3$," "$2 + 2 = 4$," and "$\sqrt{2}$ is a real number."

Special attention needs to be paid to *variables* in mathematical statements. By itself, a sentence such as "$x + 2 = 5$" is not a mathematical statement, because without further information about x, it does not have a definite truth value. A sentence containing one or more variables that have not been assigned specific values is called an ***open sentence*** or a ***predicate***.

There are two ways to turn an open sentence into a mathematical statement. One is to attach a ***quantifier*** to it. This is a phrase such as "for all x" or "there exists an x," which specifies which values of x are supposed to make the statement true. We will discuss quantifiers in much more detail later in the course.

A simpler way to make an open sentence into a mathematical statement is to put it in a context in which each of the variables has been assigned a definite meaning.

The simplest way to do that is just to assign a specific value to a variable, as in "Let $x = 2$." Once that is done, the symbol x can be used thereafter with the same meaning as 2, and a sentence like $x + 2 = 5$ is a perfectly good mathematical statement (a false one, in this case).

It is also often useful to assign a less specific, but nonetheless definite, meaning to a variable. Some examples are "Let x be a real number whose square is 2," or "Let n be an arbitrary integer." From then on, the variable can be treated as having a definite meaning, even though we may not know what it is.

When we discuss the grammar of mathematical statements, we will often have occasion to use examples involving variables that have not been previously assigned meanings. When variables appear in such a context, our standard agreement will be to assume that the variables have already been assigned definite meanings unless otherwise specified. Without this agreement, the sentence "If $x \neq 0$, then $x^2 > 0$" would not qualify as a statement because it has a variable, x, that has not been assigned a value. With this agreement, we can call "If $x \neq 0$, then $x^2 > 0$ a statement, and may discuss, for instance, the *converse* of this statement.