Math 300 **Introduction to Mathematical Reasoning** **Fall 2009**

The Real Numbers and the Integers

UNDEFINED TERMS

To avoid circularity, we cannot give every term a rigorous mathematical definition; we have to accept some things as undefined terms. For this course, we will take the following fundamental notions as undefined terms. You already know what these terms mean; but the only facts about them that can be used in proofs are the ones expressed in the axioms listed below (and any theorems that can be proved from the axioms).

- $\boxed{\text{Real number:}}$ Intuitively, a real number represents a point on the number line, or a (signed) distance left or right from the origin, or any quantity that has a finite or infinite decimal representation. Real numbers include integers, positive and negative fractions, and irrational numbers like π, e, and $\sqrt{2}$.
- $\boxed{\text{Integer:}}$ Intuitively, an integer is a whole number (positive, negative, or zero).
- $\boxed{\text{Zero:}}$ The number zero is denoted by $\boxed{0}$.
- $\boxed{\text{One:}}$ The number one is denoted by $\boxed{1}$.
- $\boxed{\text{Sum:}}$ The sum of two real numbers a and b is denoted by $\boxed{a+b}$.
- $\boxed{\text{Product:}}$ The product of two real numbers a and b is denoted by $\boxed{ab}$ or $\boxed{a \cdot b}$ or $\boxed{a \times b}$.
- $\boxed{\text{Less than:}}$ To say that a is less than b, denoted by $\boxed{a < b}$, means intuitively that a is to the left of b on the number line.

DEFINITIONS

In all the definitions below, a and b represent arbitrary real numbers.

- The set of all real numbers is denoted by $\boxed{\mathbb{R}}$, and the set of all integers is denoted by $\boxed{\mathbb{Z}}$.
- Real numbers a and b are said to be $\boxed{\text{equal}}$, denoted by $\boxed{a = b}$, if they are the same number.
- The numbers $\boxed{2}$ through $\boxed{9}$ are defined by $2 = 1 + 1$, $3 = 2 + 1$, etc. The decimal representations for other numbers are defined by the usual rules of decimal notation: For example, 23 is defined to be $2 \cdot 10 + 3$, etc.
- The $\boxed{\text{additive inverse}}$ or $\boxed{\text{negative}}$ of a is the number $-a$ that satisfies $a + (-a) = 0$, and whose existence and uniqueness are guaranteed by Axiom 9.
- The $\boxed{\text{difference between } a \text{ and } b}$, denoted by $\boxed{a - b}$, is the real number defined by $a - b = a + (-b)$.
- If $a \neq 0$, the $\boxed{\text{multiplicative inverse}}$ or $\boxed{\text{reciprocal}}$ of a is the number a^{-1} that satisfies $a \times a^{-1} = 1$, and whose existence and uniqueness are guaranteed by Axiom 10.
- If $b \neq 0$, the $\boxed{\text{quotient of } a \text{ and } b}$, denoted by $\boxed{a/b}$, is the real number defined by $a/b = ab^{-1}$.
- A real number is said to be $\boxed{\text{rational}}$ if it is equal to p/q for some integers p and q with $q \neq 0$. The set of all rational numbers is denoted by $\mathbb{Q}$.
- A real number is said to be $\boxed{\text{irrational}}$ if it is not rational.
- The statement $\boxed{a \text{ is less than or equal to } b}$, denoted by $\boxed{a \leq b}$, means $a < b$ or $a = b$.
- The statement $\boxed{a \text{ is greater than } b}$, denoted by $\boxed{a > b}$, means $b < a$.

- The statement $\boxed{a \text{ is greater than or equal to } b}$, denoted by $\boxed{a \geq b}$, means $a > b$ or $a = b$.
- A real number a is said to be $\boxed{\text{positive}}$ if $a > 0$.
- A real number a is said to be $\boxed{\text{negative}}$ if $a < 0$.
- A real number a is said to be $\boxed{\text{nonnegative}}$ if $a \geq 0$.
- A real number a is said to be $\boxed{\text{nonpositive}}$ if $a \leq 0$.
- If m and n are integers, we say that $\boxed{n \text{ divides } m}$, or $\boxed{m \text{ is divisible by } n}$, or $\boxed{m \text{ is a multiple of } n}$, denoted by $\boxed{n|m}$, if there is an integer k such that $m = nk$.
- An integer n is said to be $\boxed{\text{even}}$ if it is divisible by 2.
- An integer n is said to be $\boxed{\text{odd}}$ if it is not even.
- For any real number a, the $\boxed{\text{absolute value of } a}$, denoted by $|a|$, is defined by

$$|a| = \begin{cases} a & \text{if } a \geq 0, \\ -a & \text{if } a < 0. \end{cases}$$

- The $\boxed{\text{square}}$ of a is the number $a^2 = a \times a$.
- If S is a set of real numbers, a real number b is said to be an $\boxed{\text{upper bound for } S}$ if $b \geq x$ for every x in S. It is said to be a $\boxed{\text{least upper bound for } S}$ if every other upper bound b' for S satisfies $b' \geq b$. The terms $\boxed{\text{lower bound}}$ and $\boxed{\text{greatest lower bound}}$ are defined similarly.

AXIOMS

We assume that the following statements are true.

1. (Containment) Every integer is a real number.
2. (Closure of $\mathbb{Z}$) If a and b are integers, then $a + b$ and ab are integers that depend only on a and b.
3. (Closure of $\mathbb{R}$) If a and b are real numbers, then $a + b$ and ab are real numbers that depend only on a and b.
4. (Commutativity) $a + b = b + a$ and $ab = ba$ for all real numbers a and b.
5. (Associativity) $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$ for all real numbers a, b, and c.
6. (Distributivity) $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$ for all real numbers a, b, and c.
7. (Zero) 0 is an integer that satisfies $a + 0 = a = 0 + a$ for every real number a.
8. (One) 1 is an integer that is not equal to zero and satisfies $a \times 1 = a = 1 \times a$ for every real number a.
9. (Additive inverses) If a is any real number, there is a unique real number $-a$ such that $a + (-a) = 0$. If a is an integer, then so is $-a$.
10. (Multiplicative inverses) If a is any nonzero real number, there is a unique real number a^{-1} such that $a \times a^{-1} = 1$.
11. (Trichotomy law) If a and b are real numbers, then one and only one of the following three possibilities is true: $a < b$, $a = b$, or $a > b$.
12. (Addition law for inequalities) Suppose a, b, and c are real numbers. If $a < b$, then $a + c < b + c$.
13. (Multiplication law for inequalities) Suppose a, b, and c are real numbers, and $a < b$. If $c > 0$, then $ac < bc$. If $c < 0$, then $ac > bc$.
14. (Transitive law for inequalities) Suppose a, b, and c, are real numbers. If $a < b$ and $b < c$, then $a < c$.
15. (The well ordering axiom) Every nonempty set of positive integers has a smallest element.
16. (The completeness axiom) If S is any nonempty set of real numbers and S has an upper bound, then S has a least upper bound.

SELECTED THEOREMS

These theorems can be proved from the axioms in the order listed below.

1. Properties of Equality
   (a) (Reflexivity) If a is any real number, then $a = a$.
   (b) (Symmetry) If $a = b$, then $b = a$.
   (c) (Transitivity) If $a = b$ and $b = c$, then $a = c$.
   (d) If $a = b$, then $-a = -b$.
   (e) If $a = b$, and a and b are both nonzero, then $a^{-1} = b^{-1}$.
   (f) If $a = b$ and $c = d$, then $a + c = b + d$, $ac = bd$, and $a - c = b - d$.
   (g) If $a = b$ and $c = d$, and c and d are both nonzero, then $a/c = b/d$.
2. Properties of zero
   (a) $0 \times a = 0$.
   (b) If $ab = 0$, then $a = 0$ or $b = 0$.
3. Properties of signs
   (a) $-0 = 0$.
   (b) $-(-a) = a$.
   (c) $-a = (-1)a$.
   (d) $(-a)b = -(ab) = a(-b)$.
   (e) $(-a)(-b) = ab$.
4. Extensions of the distributive property
   (a) $-(a + b) = (-a) + (-b) = -a - b$.
   (b) $-(a - b) = b - a$.
   (c) $-(-a - b) = a + b$.
   (d) $a + a = 2a$.
   (e) $a(b - c) = ab - ac = (b - c)a$.
   (f) $(a + b)(c + d) = ac + ad + bc + bd$.
   (g) $(a + b)(c - d) = ac - ad + bc - bd = (c - d)(a + b)$.
   (h) $(a - b)(c - d) = ac - ad - bc + bd$.
5. Properties of inverses
   (a) $1^{-1} = 1$.
   (b) $(a^{-1})^{-1} = a$ if a is nonzero.
   (c) $(-a)^{-1} = -(a^{-1})$ if a is nonzero.
   (d) $(ab)^{-1} = a^{-1}b^{-1}$ if a and b are nonzero.
   (e) $(a/b)^{-1} = b/a$ if a and b are nonzero.
6. Properties of quotients
   (a) $a/1 = a$.
   (b) $(a/b)(c/d) = (ac)/(bd)$ if b and d are nonzero.
   (c) $(a/b)/(c/d) = (ad)/(bc)$ if b, c, and d are nonzero.
   (d) $(ac)/(bc) = a/b$ if b and c are nonzero.
   (e) $(-a)/b = -(a/b) = a/(-b)$ if b is nonzero.
   (f) $(-a)/(-b) = a/b$ if b is nonzero.
   (g) $a/b + c/d = (ad + bc)/(bd)$ if b and d are nonzero.
   (h) $a/b - c/d = (ad - bc)/(bd)$ if b and d are nonzero.
7. Properties of inequalities
   (a) $0 < 1$.
   (b) If $a < b$, then $-a > -b$.
   (c) If $a < b$ and a and b are both positive, then $a^{-1} > b^{-1}$.
   (d) If $a \leq b$ and $b \leq c$, then $a \leq c$.
   (e) If $a \leq b$ and $b < c$, then $a < c$.
   (f) If $a < b$ and $b \leq c$, then $a < c$.

(g) If $a < b$ and $c < d$, then $a + c < b + d$.
(h) If $a \leq b$ and $c < d$, then $a + c < b + d$.
(i) If $a \leq b$ and $c \leq d$, then $a + c \leq b + d$.
(j) If $a < b$ and $c > 0$, then $ac < bc$.
(k) If $a < b$ and $c < 0$, then $ac > bc$.
(l) If $a \leq b$ and $c > 0$, then $ac \leq bc$.
(m) If $a \leq b$ and $c < 0$, then $ac \geq bc$.
(n) If $a < b$ and $c < d$, and a, b, c, d are nonnegative, then $ac < bd$.
(o) If $a \leq b$ and $c \leq d$, and a, b, c, d are nonnegative, then $ac \leq bd$.
(p) $ab > 0$ if and only if a and b are both positive or both negative.
(q) $ab < 0$ if and only if one is positive and the other is negative.
(r) If $a \leq b$ and $b \leq a$, then $a = b$.
(s) There does not exist a smallest positive real number.
(t) (Density) If $a < b$, then there is a real number x such that $a < x < b$.

8. Properties of squares
   (a) For every a, $a^2 \geq 0$.
   (b) $a^2 = 0$ if and only if $a = 0$.
   (c) $(-a)^2 = a^2$.
   (d) $(a^{-1})^2 = 1/a^2$.
   (e) If a and b are positive, then $a < b \Rightarrow a^2 < b^2$.
   (f) If a and b are negative, then $a < b \Rightarrow a^2 > b^2$.
9. Properties of absolute values
   (a) For every a, $|a| \geq 0$.
   (b) $|a| = 0$ if and only if $a = 0$.
   (c) $|-a| = |a|$.
   (d) $|a^{-1}| = 1/|a|$ if $a \neq 0$.
   (e) $|ab| = |a|\,|b|$.
   (f) $|a/b| = |a|/|b|$ if $b \neq 0$.
   (g) (The triangle inequality) $|a + b| \leq |a| + |b|$.
   (h) If a and b are both nonnegative, then $|a| \geq |b|$ if and only if $a \geq b$.
   (i) If a and b are both negative, then $|a| \geq |b|$ if and only if $a \leq b$.
10. Order properties of integers
   (a) 1 is the smallest positive integer.
   (b) If m and n are integers such that $m > n$, then $m \geq n + 1$.
   (c) There does not exist a largest or smallest integer.
11. Laws of exponents
   In statements (a)–(d) *below, m and n are assumed to be nonnegative integers.*
   (a) $a^n b^n = (ab)^n$.
   (b) $a^{m+n} = a^m a^n$.
   (c) $(a^m)^n = a^{mn}$.
   (d) $a^n / b^n = (a/b)^n$ if b is nonzero.
   (e) Identities 11(a)–11(d) hold for arbitrary integers m and n, provided a and b are nonzero.
12. Divisibility properties of integers
   In each of the following statements, m, n, and p are assumed to be integers.
   (a) If $m|n$ and $n|p$, then $m|p$.
   (b) If $m|n$ and $m|p$, then $m|(n + p)$.
   (c) If $m|n$ or $m|p$, then $m|np$.
   (d) n is even if and only if $n = 2k$ for some integer k.
   (e) n is odd if and only if $n = 2k + 1$ for some integer k.
   (f) n^2 is even if and only if n is even.
   (g) n^2 is odd if and only if n is odd.
   (h) $m + n$ is even if and only if m and n are both even or both odd.
   (i) mn is even if and only if either m or n is even.