Math 135 **Honors Calculus** **Winter 2017**

Handout 1: L'Hôpital's Rule, the ∞/∞ case

Theorem 1. *Suppose f and g are continuous on $[a,b]$ and differentiable on (a,b), $g'(x)$ is never zero for $x \in (a,b)$, and $f(x) \to +\infty$ and $g(x) \to +\infty$ as $x \to a^+$. If*

$$\frac{f'(x)}{g'(x)} \to L \text{ as } x \to a^+,$$

then

$$\frac{f(x)}{g(x)} \to L \text{ as } x \to a^+.$$

Proof. Let $\varepsilon > 0$ be given. Because $f'(x)/g'(x) \to L$, there exists some $\delta_1 > 0$ such that if $a < x < a + \delta_1$, then

$$\left| \frac{f'(x)}{g'(x)} - L \right| < \frac{\varepsilon}{4}. \tag{1}$$

For simplicity, set $c = a + \delta_1$.

Since both $f(x) \to +\infty$ and $g(x) \to +\infty$ as $x \to a^+$, we can choose $\delta_2 > 0$ such that $f(x) > \max(f(c), 0)$ and $g(x) > \max(g(c), 0)$ whenever $a < x < a + \delta_2$. For any such x, note that

$$\frac{f(x)}{g(x)} = \left(\frac{f(c) - f(x)}{g(c) - g(x)} \right) \left(\frac{f(x)}{f(c) - f(x)} \cdot \frac{g(c) - g(x)}{g(x)} \right). \tag{2}$$

Let's denote the expression inside the second set of parentheses as $H(x)$. A little algebraic manipulation shows that

$$H(x) = \frac{\left(\frac{g(c)}{g(x)} - 1 \right)}{\left(\frac{f(c)}{f(x)} - 1 \right)}.$$

Because $g(c)/g(x) \to 0$ and $f(c)/f(x) \to 0$, it follows that $H(x) \to 1$ as $x \to a^+$. Therefore, we can choose $\delta_3 > 0$ such that if $a < x < a + \delta_3$, then

$$|H(x) - 1| < \min\left(\frac{\varepsilon}{2|L| + 1}, 1 \right).$$

In particular, this implies

$$|H(x)| = |H(x) - 1 + 1| \leq |H(x) - 1| + |1| < 1 + 1 = 2, \tag{3}$$

and

$$|L|\,|H(x) - 1| \leq |L| \frac{\varepsilon}{2|L| + 1} \leq \frac{\varepsilon}{2}. \tag{4}$$

Let $\delta = \min(\delta_1, \delta_2, \delta_3)$, and suppose x is any number such that $a < x < \delta$. We can apply the Cauchy mean-value theorem (Theorem 11.5.2 in the textbook) on the interval $[x, c]$, showing that there is some $r \in (x, c)$ such that

$$\frac{f'(r)}{g'(r)} = \frac{f(c) - f(x)}{g(c) - g(x)}.$$

Since this r satisfies $a < x < r < c = a + \delta_1$, (1) implies that

$$\left| \frac{f'(r)}{g'(r)} - L \right| < \frac{\varepsilon}{4}.$$

Therefore, using (2), (3), and (4), we obtain

$$\begin{aligned}
\left|\frac{f(x)}{g(x)} - L\right| &= \left|\left(\frac{f(c)-f(x)}{g(c)-g(x)}\right) H(x) - L\right| \\
&= \left|\frac{f'(r)}{g'(r)} H(x) - L\right| \\
&= \left|\frac{f'(r)}{g'(r)} H(x) - LH(x) + LH(x) - L\right| \\
&\le \left|\frac{f'(r)}{g'(r)} H(x) - LH(x)\right| + |LH(x) - L| \\
&= |H(x)| \left|\frac{f'(r)}{g'(r)} - L\right| + |L|\ |H(x) - 1| \\
&\le 2 \left|\frac{f'(r)}{g'(r)} - L\right| + |L|\ |H(x) - 1| \\
&\le 2\,\frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

This completes the proof. □

There are other versions of the theorem: The limit L can be $\pm\infty$, or the approach can be any one of the following instead of $x \to a^+$:

$$x \to a^-, \quad x \to a, \quad x \to +\infty, \quad x \to -\infty.$$

These variations can be handled by minor modifications to the proof, just as in the 0/0 case in the textbook.