Handout 3: The Complex Numbers[1] (Revised 2/28/2017)

1. The Complex Plane

The intuitive idea of the complex numbers is that we enlarge the set of real numbers by introducing a new number i that satisfies $i^2 = -1$, and then consider all numbers of the form $x + iy$, where x and y are real. The rules for addition and multiplication should be more or less what you would expect, assuming that i follows the usual rules of algebra:

$$\begin{aligned} (x+iy)+(u+iv) &= (x+u)+i(y+v), \\ (x+iy)(u+iv) &= xu + iyu + ixv + i^2yv = (xu-yv)+i(yu+xv). \end{aligned} \tag{1}$$

Formally, we define a ***complex number*** to be an ordered pair of real numbers (x, y). Motivated by the heuristic calculations above, we define addition and multiplication of complex numbers by

$$(x,y)+(u,v) = (x+u, y+v), \qquad (x,y)(u,v) = (xu-yv,\ yu+xv).$$

The set of all complex numbers, called the ***complex plane***, is denoted by $\mathbb{C}$; as a *set*, it is the same as $\mathbb{R}^2$, but with additional operations defined on it.

For any real number x, when we're working in the complex number system we typically use x as a shorthand notation for the complex number $(x, 0)$, and thereby consider $\mathbb{R}$ as a subset of $\mathbb{C}$. Note that the addition and multiplication rules for these "real complex numbers" correspond to ordinary addition and multiplication in $\mathbb{R}$:

$$(x,0)+(u,0) = (x+u,0), \qquad (x,0)(y,0) = (xy,0).$$

We define i to be the complex number $(0, 1)$, and note that it satisfies

$$i^2 = (0,1)(0,1) = (0\cdot 0 - 1\cdot 1,\ 1\cdot 0 + 0\cdot 1) = (-1,0) = -1.$$

Thus any complex number (x, y) can also be written as $x + iy$, because the latter is shorthand for

$$(x,0)+(0,1)(y,0) = (x,0)+(0\cdot y - 1\cdot 0,\ 1\cdot y + 0\cdot 0) = (x,y).$$

With this understanding, we will almost always write complex numbers in the form $x+iy$ rather than (x, y), and complex addition and multiplication actually do obey the rules in (1). Note that the definition implies that $x + iy = u + iv$ if and only if $x = u$ and $y = v$. Typically we use the letters w and z to represent individual complex numbers, with $w = u + iv$ and $z = x + iy$. A complex number of the form $iy = 0 + iy$ for $y \in \mathbb{R}$ is called an ***imaginary number*** (or sometimes ***pure imaginary***). The term "imaginary" is a historical holdover—it took mathematicians some time to accept the fact that i (for "imaginary," naturally) was a perfectly good mathematical object. Electrical engineers (who make heavy use of complex numbers) reserve the letter i to denote electric current and they use j for the complex number $(0, 1)$.

Proposition 1. *Complex addition and multiplication satisfy the commutative, associative, and distributive laws. The number $0 = 0 + 0i$ is an additive identity, and $1 = 1 + 0i$ is a multiplicative identity. Every complex number has an additive inverse, and every nonzero complex number has a multiplicative inverse. Thus the complex numbers form a field.*

Proof. Commutativity and associativity of addition are easy to check, as are commutativity of multiplication and the fact that 0 and 1 function as identities for addition and multiplication. The additive inverse of $x+iy$ is $(-x)+i(-y)$, as you can quickly verify. Associativity of multiplication and the distributive law connecting addition and multiplication are straightforward but tedious calculations and are left as exercises.

The only part of the proposition that is not quite straightforward is the existence of multiplicative inverses. Suppose $x + iy$ is a nonzero complex number. Note that $(x + iy)(x - iy) = x^2 + iyx - ixy - i^2y^2 = x^2 + y^2 \neq 0$. Reasoning heuristically for a moment, we carry out the following computation to "rationalize the denominator" and derive an expression for $(x + iy)^{-1}$:

$$\frac{1}{x+iy} = \frac{1}{x+iy}\,\frac{x-iy}{x-iy} = \frac{x-iy}{(x+iy)(x-iy)} = \left(\frac{x}{x^2+y^2}\right) + i\left(\frac{-y}{x^2+y^2}\right). \tag{2}$$

[1] Written by Jack Lee, based on earlier notes by Bob Phelps and Tom Duchamp.

This last expression is a well-defined complex number because $x^2 + y^2 \neq 0$, and you can check by direct computation that its product with $x + iy$ is equal to 1 (see Exercise 1(c) below). □

Exercises 1.

(a) Prove that complex multiplication is associative.
(b) Prove that complex addition and multiplication satisfy the distributive law.
(c) Prove that the product of $z = x + iy$ and the expression on the right-hand side of (2) equals 1.
(d) Verify each of the following:
 1. $(\sqrt{2} - i) - i(1 - \sqrt{2}i) = -2i$
 2. $\dfrac{1+2i}{3-4i} + \dfrac{2-i}{5i} = -\dfrac{2}{5}$
 3. $\dfrac{5}{(1-i)(2-i)(3-i)} = \dfrac{1}{2}i$
 4. $(1-i)^4 = -4$

(e) Find all complex numbers $z = x + iy$ such that $z^2 = 1 + i$.

Thanks to the preceding proposition, all of the theorems about real numbers that can be proved using only the field axioms are true of the complex numbers as well. However, there is no useful ordering of the complex numbers, so it doesn't make sense to say $z < w$ unless both z and w are real. In fact, it is common practice in mathematical writing to interpret the statement "$x < y$" to mean "x and y are real numbers and x is less than y."

If $z = x + iy$ with x and y real, we call x the ***real part of z*** and y the ***imaginary part***, and we write $x = \operatorname{Re} z$, $y = \operatorname{Im} z$. (*Note that the imaginary part of z is a real number!*) Thus $z_1 = z_2$ if and only if $\operatorname{Re} z_1 = \operatorname{Re} z_2$ and $\operatorname{Im} z_1 = \operatorname{Im} z_2$.

The complex numbers, being essentially ordered pairs of real numbers, can be visulalized as points in the plane (sometimes called an ***Argand diagram*** in this context). The x-axis is called the ***real axis*** and contains the set of real numbers (thought of as a subset of the complex numbers), and the y-axis is called the ***imaginary axis*** and contains all of the imaginary numbers. For example, the complex numbers $3 + 4i$ and $3 - 4i$ are illustrated in Fig. 1, and complex addition in Fig. 2.

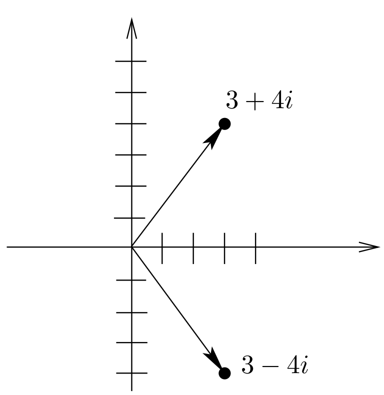

Fig. 1. Complex numbers in the plane

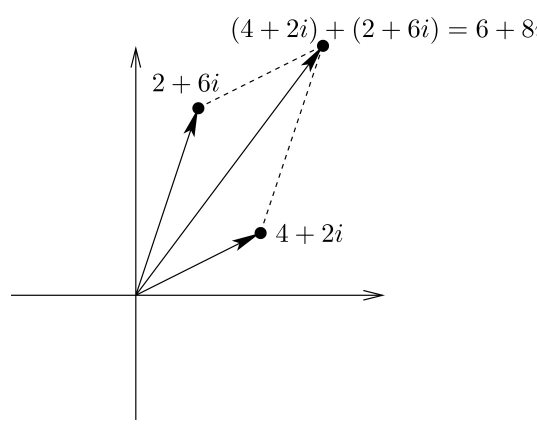

Fig. 2. Addition of complex numbers

The geometric interpretation of complex multiplication is more complicated; we will return to it later when we treat the polar representation of complex numbers.

The expression $x - iy$ appears so often and is so useful that it is given a name. It is called the ***complex conjugate*** of $z = x + iy$ and a shorthand notation for it is $\overline{z}$; that is, if $z = x + iy$, then $\overline{z} = x - iy$. For example, $\overline{3 + 4i} = 3 - 4i$, as illustrated in Fig. 1.

Another important quantity associated with a given complex number z is its ***modulus*** or ***absolute value*** $|z|$, which is the Euclidean distance from the origin to the point z:

$$|z| = \sqrt{x^2 + y^2} = \left((\operatorname{Re} z)^2 + (\operatorname{Im} z)^2\right)^{1/2}.$$

Note that $|z|$ is a *real* number. For example, $|3 + 4i| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$. This leads to the inequality

$$\operatorname{Re} z \le |\operatorname{Re} z| = \sqrt{(\operatorname{Re} z)^2} \le \sqrt{(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2} = |z|$$

Similarly, $\operatorname{Im} z \le |\operatorname{Im} z| \le |z|$.

Exercises 2.
Prove the following for all complex numbers z and w:

(a) $\overline{z + w} = \overline{z} + \overline{w}$.
(b) $\overline{zw} = \overline{z} \cdot \overline{w}$.
(c) $z\overline{z} = |z|^2$.
(d) $|\overline{z}| = |z|$.
(e) $|zw| = |z||w|$.
(f) $|cz| = c|z|$ if $c > 0$.
(g) If $z \ne 0$, then $z/|z|$ has modulus 1.
(h) $\operatorname{Re} z = (z + \overline{z})/2$.
(i) $\operatorname{Im} z = (z - \overline{z})/(2i)$.
(j) z is a real number if and only if $\overline{z} = z$.
(k) z is an imaginary number if and only if $\overline{z} = -z$.

2. Complex-valued Functions

We will sometimes have occasion to discuss complex-valued functions of a real variable. (Complex-valued functions of a *complex variable* constitute a whole different subject, called ***complex analysis***, which we will not treat.) In general, a complex-valued function $w = w(t)$ of the real variable t can be written

$$w(t) = u(t) + iv(t)$$

where u and v are real-valued functions. A complex-valued function can be thought of a defining a parametrized curve in the complex plane.

Given such a function $w(t) = u(t) + iv(t)$, we say w is ***differentiable*** if both u and v are, and we define ***the derivative of $w(t)$ with respect to t*** to be the function

$$w'(t) = u'(t) + iv'(t).$$

Proposition 2. *If $w = w(t)$ and $z = z(t)$ are differentiable complex-valued functions of a real variable, then the following formulas hold:*

(a) $w' = 0$ *if w is a constant curve,* $w(t) \equiv C$.
(b) $(z + w)' = z' + w'$.
(c) $(zw)' = z'w + zw'$.
(d) $(az)' = az'$ *for* $a \in \mathbb{R}$.
(e) $(z^n)' = nz^{n-1}z'$ *for* $n \in \mathbb{Z}^+$.
(f) $(1/z)' = -z'/z^2$ *wherever* $z \ne 0$.

Exercises 3. Prove the preceding proposition by expanding the left- and right-hand sides of each identity in terms of real and imaginary parts.

3. The Complex Exponential Function

Next, we'd like to make sense of the exponential function e^z when z is a complex number. Writing $z = x + iy$, we would expect that

$$e^z = e^{x+iy} = e^x e^{iy}.$$

We already know how to interpret e^x, so the only new thing we have to make sense of is e^{iy} when y is a real number.

Here are two approaches to making sense of this expression. First, recall the power series expansion for e^x when x is real:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

Just computing formally for the moment, without worrying about convergence, we might try substituting $x = it$ to get

$$e^{it} = \sum_{k=0}^{\infty} \frac{(it)^k}{k!} = 1 + it + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \dots + \frac{(it)^n}{n!} + \dots.$$

Note that even powers of i are all real, because $i^{2j} = (i^2)^j = (-1)^j$, and odd powers have an extra factor of i and are thus all imaginary. Let us separate the even and odd terms in the series, writing $k = 2j$ in the even case and $k = 2j+1$ in the odd case. This gives

$$\begin{aligned} e^{it} &= \sum_{j=0}^{\infty} \frac{(it)^{2j}}{(2j)!} + \sum_{j=0}^{\infty} \frac{(it)^{2j+1}}{(2j+1)!} \\ &= \sum_{j=0}^{\infty} \frac{(-1)^j t^{2j}}{(2j)!} + i \sum_{j=0}^{\infty} \frac{(-1)^j t^{2j+1}}{(2j+1)!}. \end{aligned}$$

But these last two series should be familiar: They are the series expansions for $\cos t$ and $\sin t$, respectively! Therefore, we are led to guess that a reasonable interpretation for e^{it} might be

$$e^{it} = \cos t + i \sin t. \tag{3}$$

Here's another way to approach the question. We can hope that, however we decide to define e^{it}, it should obey the usual rules of calculus. Let's separate out the real and imaginary parts of e^{it} and write them as follows:

$$e^{it} = u(t) + iv(t),$$

where u and v are real-valued functions of the real variable t. Assuming that all the usual rules of calculus apply when we differentiate a complex-valued function with respect to t, we just blithely differentiate both sides of this equation to find

$$ie^{it} = u'(t) + iv'(t).$$

The left-hand side of this equation is $i\big(u(t) + iv(t)\big) = -v(t) + iu(t)$, so equating real and imaginary parts yields

$$\begin{aligned} u'(t) &= -v(t), \\ v'(t) &= u(t). \end{aligned}$$

Assuming u is twice differentiable, this in turn implies $u''(t) = -v'(t) = -u(t)$. We also should expect that $e^{i0} = 1 = 1 + 0i$, which implies that $u(0) = 1$ and $v(0) = 0$, and therefore from the equations above we conclude that $u'(0) = 0$. Thus, if there's any justice in the world, u should be a solution to the following initial-value problem:

$$\begin{aligned} u''(t) + u(t) &= 0, \\ u(0) &= 1, \\ u'(0) &= 0. \end{aligned}$$

It is not hard to guess that a solution to this problem is $u(t) = \cos t$, and the existence and uniqueness theorem guarantees that it is the only solution. This in turn implies $v(t) = -u'(t) = \sin t$. Once again, we are let to expect that $e^{it} = \cos t + i \sin t$ should be a reasonable interpretation of e^{it}.

Motivated by these computations, let us simply *define* e^{it} for real t by the following formula, known as ***Euler's formula:***

$$e^{it} = \cos t + i \sin t.$$

Then we extend the definition to arbitrary complex exponents by means of the following formula for any $z = x + iy \in \mathbb{C}$:

$$e^z = e^x e^{iy} = e^x(\cos y + i \sin y).$$

For example,

$$e^{2\pi i} = +1, \quad e^{i\pi/2} = i, \quad e^{i\pi/6} = \sqrt{3} + i,$$

and one of the most elegant and intriguing equations in all of mathematics,

$$e^{i\pi} = -1, \qquad \text{or equivalently,} \qquad e^{i\pi} + 1 = 0.$$

The second version ties together the five most fundamental constants in mathematics (0, 1, e, i, and π), the three most fundamental operations (addition, multiplication, and exponentiation), and the single most important relation (equality). It makes a compelling case for the unity and beauty of mathematics.

Exercises 4. Prove the following facts for all $z, w \in \mathbb{C}$:

(a) $\overline{e^z} = e^{\overline{z}}$.
(b) $|e^z| = e^{\operatorname{Re} z}$.
(c) $1/(e^z) = e^{-z}$.
(d) $e^{z+w} = e^z e^w$.
(e) [deleted]
(f) $\dfrac{d}{dt}\big(e^{zt}\big) = ze^{zt}$ for all $t \in \mathbb{R}$.

4. Polar Representation of Complex Numbers

Recall that we can represent points in the plane using polar coordinates as well as rectangular coordinates. The relation between the rectangular coordinates (x, y) and the polar coordinates $[r, \theta]$ is

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

Thus for the complex number $z = x + iy$, we can write

$$z = r(\cos\theta + i\sin\theta).$$

Using the complex exponential function introduced above, this can be written in the more compact form

$$z = re^{i\theta}.$$

It follows from Exercise 4(b) above that $|e^{i\theta}| = 1$, so we can recover r and θ by setting $r = |re^{i\theta}| = |z|$ and choosing θ to be any angle such that $\cos\theta = x/r$ and $\sin\theta = y/r$. (When working with polar coordinates in the complex plane, we will always insist that $r \geq 0$.) For example, the complex number $z = 2\sqrt{3} + 2i$ can also be written as $2e^{i\pi/6}$, as illustrated in Fig. 3.

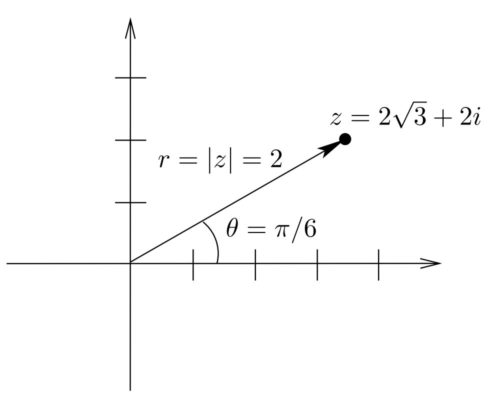

Fig. 3. Polar coordinates in $\mathbb{C}$

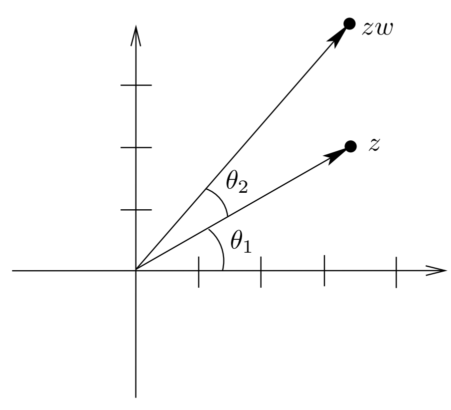

Fig. 4. Multiplication of complex numbers

The conditions for equality of two complex numbers using polar coordinates are not quite as simple as they were for rectangular coordinates. If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then $z_1 = z_2$ if and only if $r_1 = r_2$ and $\theta_1 = \theta_2 + 2\pi k$ for some $k \in \mathbb{Z}$.

The polar coordinate representation gives us an easy way to interpret complex multiplication geometrically. Note that the angle-sum formulas for sine and cosine yield the following simple formula:

$$\begin{aligned}(e^{i\theta_1})(e^{i\theta_2}) &= (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\\ &= (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)\\ &= \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\\ &= e^{i(\theta_1+\theta_2)}.\end{aligned}$$

Thus if $z = r_1 e^{i\theta_1}$ and $w = r_2 e^{i\theta_2}$, then

$$zw = r_1 r_2 e^{i(\theta_1+\theta_2)}.$$

(See Fig. 4.) In other words, multiplication by $w = r_2 e^{i\theta_2}$ has the geometric effect of scaling the modulus by $|w|$ and rotating the direction through an angle θ_2 (counterclockwise if $\theta_2 > 0$, and clockwise if $\theta_2 < 0$).

An easy inductive argument shows that

$$\text{If } z = re^{i\theta}, \text{ then } z^n = r^n e^{in\theta}.$$

This makes it easy to solve equations like $z^3 = 1$. Indeed, writing the unknown number z as $re^{i\theta}$, we have $r^3 e^{i3\theta} = 1 = e^{0i}$, hence $r^3 = 1$ (so $r = 1$) and $3\theta = 2k\pi$ for some $k \in \mathbb{Z}$. It follows that $\theta = 2k\pi/3$, $k \in \mathbb{Z}$. There are only three distinct complex numbers of the form $e^{2k\pi i/3}$, namely $e^0 = 1$, $e^{2\pi i/3} = -\sqrt{3} + \frac{1}{2}i$, and $e^{4\pi i/3} = -\sqrt{3} - \frac{1}{2}i$. The following figure illustrates $z = 8i = 8e^{i\pi/2}$ and its three cube roots $z_1 = 2e^{i\pi/6}$, $z_2 = 2e^{5i\pi/6}$, $z_3 = 2e^{9i\pi/6}$:

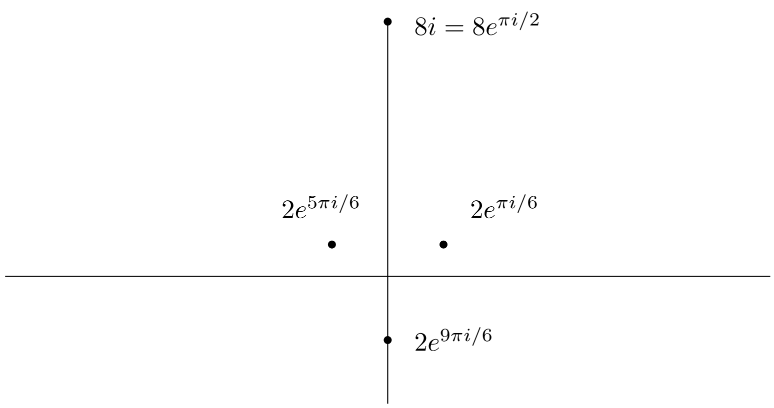

FIG. 5: The cube roots of $8i$

From the fact that $(e^{i\theta})^n = e^{in\theta}$ we obtain De Moivre's formula:

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

By expanding on the left and equating real and imaginary parts, we can deduce trigonometric identities that can be used to express $\cos n\theta$ and $\sin n\theta$ as a sum of terms of the form $(\cos\theta)^j(\sin\theta)^k$. For example, taking $n = 2$ one gets $\cos 2\theta = \cos^2\theta - \sin^2\theta$. For $n = 3$ one gets $\cos 3\theta = \cos^3\theta - \cos\theta\sin^2\theta - 2\sin^2\theta\cos^2\theta$.

Exercises 5.

(a) Let $z_1 = 3i$ and $z_2 = 2 - 2i$
  1. Plot the points $z_1 + z_2$, $z_1 - z_2$ and $\overline{z_2}$.
  2. Compute $|z_1 + z_2|$ and $|z_1 - z_2|$.
  3. Express z_1 and z_2 in polar form.

(b) Let $z_1 = 6e^{i\pi/3}$ and $z_2 = 2e^{-i\pi/6}$. Plot z_1, z_2, $z_1 z_2$ and z_1/z_2.

(c) Find all complex numbers z that satisfy $z^3 = -1$.

(d) Find all complex numbers $z = re^{i\theta}$ such that $z^2 = \sqrt{2}e^{i\pi/4}$.